Stochastic processes and derivatives
Sheet 7

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Exercise 1 (Stochastic optimal control)**.** The market offers a (possibly random) bounded rate of interest r_t at time $t \geq 0$ on a d-dimensional financial model

$$\frac{\mathrm{d}S_t^{(i)}}{S_t^{(i)}} \quad = \quad b_t^{(i)}\,\mathrm{d}t \;+\; \sum_{j=1}^{d} \sigma_t^{(i,j)}\,\mathrm{d}W_t^{(j)}$$

b and σ are (possibly) random processes, the volatility matrix σ_t is invertible and both σ_t and b_t are bounded for all t a.s. We assume also that the market price of risk $\lambda_t := \sigma_t^{-1}(b_t - r_r\mathbf{1})$ is bounded. Define a utility function $U :]0,\infty[\to \mathbb{R}$ to be a continuously differentiable, strictly increasing, strictly concave function s.t. $U'(x) \xrightarrow[x\to\infty]{} 0$ and $U'(x) \xrightarrow[\substack{x\to 0\\ >}]{} \infty$.

1. Show that $U_{\log}(x) = \log x$ and $U_\alpha(x) = \frac{x^\alpha}{\alpha}$, $\alpha \in]0,1[$ are utility functions.

Fix an initial wealth $x \in]0,\infty[$. For any strategy $\delta \in \mathcal{A}$ (the set of admissible strategies), let $(X_t^{(\delta,x)})_{t\geq 0}$ denote the value of the self-financing portfolio with the strategy δ with initial wealth x. We will show that there exists an admissible strategy which maximizes the expected utility *i.e.* $\exists! \delta^* \in \mathcal{A}$ s.t.

$$\mathbb{E}\left[U(X_T^{(\delta^*,x)})\right] \quad = \quad \max_{\delta\in\mathcal{A}} \mathbb{E}\left[U\left(X_T^{(\delta,x)}\right)\right]. \tag{0.1}$$

2. (Legendre–Fenchel transform) Show that the inverse function of $U'(\cdot)$, denoted by $I :]0,\infty[\to]0,\infty[$, exists and is strictly decreasing. Prove that

$$\begin{aligned}
\tilde{U}(y) \quad &:= \quad \max_{x\in\mathbb{R}_+^*}\{U(x) \;-\; xy\} \;=\; U(I(y)) \;-\; yI(y)\;;\\
\tilde{U}'(y) \quad &= \quad -I(y)\;;\\
U(x) \quad &= \quad \tilde{U}(U'(x)) \;+\; xU'(x) \;=\; \min_{y\in\mathbb{R}_+^*}\{\tilde{U}(y) \;+\; xy\}.
\end{aligned}$$

3. (Lagrange multiplier) Recall that the price at time $t < T$ of a contingent claim on the assets S with payoff Ψ at maturity T is given by

$$X_t \quad = \quad \frac{\mathbb{E}[H_T\Psi|\mathcal{F}_t]}{H_t},$$

where H is a process determined by the market price of risk and the rate of interest. The initial value X_0 is given by $x = \mathbb{E}[H_T\Psi]$. Prove that, $\forall y > 0$,

$$\mathbb{E}[U(\Psi)] \quad \leq \quad \mathbb{E}\left[\tilde{U}(yH_T)\right] \;+\; xy.$$

Prove that this is an equality iff $\Psi = I(yH_T)$.

4. Define the mapping $\mathcal{X}_0(y) := \mathbb{E}[H_T I(yH_T)]$. Prove that $\mathcal{X}_0$ is a strictly decreasing function with

$$\mathcal{X}_0(y) \xrightarrow[\substack{y\to 0\\ >}]{} \infty\,, \qquad \mathcal{X}_0(y) \xrightarrow[y\to\infty]{} 0.$$

5. Prove that an optimal investing strategy is linked to a selt financing hedging strategy such that the expected utility of the portfolio at maturity T is equal to the optimal expected utility given in (0.1).

6. Compute explicitly an optimal strategy in the setting of logarithmic utility $U_{\log}$ and the power utility U_α, for the power utility assume r, b and σ are deterministic. To simplify the computations take $d = 1$.

Proof. 1. Trivial.

2. U' is strictly decreasing and $U'(x) > 0$ as U is strictly concave and increasing. Hence the inverse exists and takes value in $]0,\infty[\to]0,\infty[$.

   We have $\frac{\mathrm{d}}{\mathrm{d}x^2}(U(x) - xy) = U''(x) < 0$ as U' is strictly decreasing, then the max exists and is reached in x_0 s.t.

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}x}_{|x=x_0}(U(x) - xy) \ = \ 0 \\ \Leftrightarrow \quad & U'(x_0) - y \ = \ 0 \\ \Leftrightarrow \quad & x_0 \ = \ I(y). \end{aligned}$$

   And then $\tilde{U}(y) = U(I(y)) - yI(y)$. We verify easily the second and third assertions.

3. With the precedent question:

$$\begin{aligned} \mathbb{E}\left[U(\Psi)\right] \ & \triangleq \ \mathbb{E}\left[\min_{\tilde{y}\in\mathbb{R}_+^*}\{\tilde{U}(\tilde{y}) + \Psi\tilde{y}\}\right] \\ & \leq \ \mathbb{E}\left[\tilde{U}(\tilde{y}) + \Psi\tilde{y}\right], \qquad \forall\tilde{y} > 0 \\ & \leq \ \mathbb{E}\left[\tilde{U}(H_T y)\right] \ + \ \mathbb{E}\left[\Psi H_T y\right], \qquad \text{with } y := \frac{\tilde{y}}{H_T} \\ & \leq \ \mathbb{E}\left[\tilde{U}(H_T y)\right] \ + \ xy. \end{aligned}$$

   And the precedent question tells us that we have the equality for $U'(\Psi) = yH_T \ \Leftrightarrow \ \Psi = I(yH_T)$.

4. As U' is decreasing it is easy to see that $\mathcal{X}_0$ is decreasing. Moreover,

$$\begin{aligned} \lim_{\substack{y\to 0\\ >}} \mathcal{X}_0(y) \ & \triangleq \ \lim_{\substack{y\to 0\\ >}} \mathbb{E}[H_T I(yH_T)] \\ & = \ \mathbb{E}\left[H_T \lim_{\substack{y\to 0\\ >}} I(yH_T)\right] \qquad \text{(Monotone convergence)} \\ & = \ \infty, \end{aligned}$$

   indeed $\lim_{x\to\infty} U'(x) = 0 \ \Rightarrow \ \lim_{\substack{y\to 0\\ >}} I(y) = \infty$, and $H_T > 0$. Same proof for the other limit.

5. Let's fix the initial wealth $\bar{x}$, we saw in the precedent question that $\mathcal{X}_0$ is a bijection, hence $\exists!\bar{y}$ s.t. $\mathcal{X}_0(\bar{y}) = \bar{x}$, so $\bar{x} = \mathbb{E}[H_T I(\bar{y}H_T)]$. From the question 3, a candidate for the payoff is $\bar{\Psi} = I(\bar{y}H_T)$.

   We are in a complete market so $\exists\bar{\delta} \ / \ X_T^{(\bar{x},\bar{\delta})} = \bar{\Psi}$, and define $\Psi = X_T^{(\bar{x},\delta)}$. We saw that $\forall\delta$,

$$\mathbb{E}\left[U(X_T^{(\bar{x},\delta)}\right] \ \leq \ \mathbb{E}\left[\tilde{U}(\bar{y}H_T)\right] \ + \ \bar{x}\bar{y},$$

   with equality in $X_T^{\bar{x},\bar{\delta}}$. Hence $\bar{\delta}$ is a maximizer of (0.1).

6. We clearly have $I_{\log(y)} = \frac{1}{y}$, hence for the optimal strategy δ^*,

$$\begin{aligned} X_t^{(x,\delta^*)} \ & = \ \mathbb{E}\left[\frac{H_T}{H_t} I(yH_T)\middle|\mathcal{F}_t\right] \\ & = \ \mathbb{E}\left[\frac{1}{yH_t}\right] \\ & = \ xH_t^{-1} \end{aligned}$$

□

Exercise 2 (Leveraging effect)**.** For a fixed time T, let $\alpha : [0,T[\to]0,\infty[$ be a differentiable and define the stochastic process $I_t := \int_0^t \alpha_s \,\mathrm{d}W_s$ and the deterministic process $A_t := \int_0^t \alpha_s^2 \,\mathrm{d}s$. For the fisrt part of this problem, we will study the properties of the process I_t ; we shall then use it to study financial situation with an inadmissible trading strategy.

1. For any given t, what is the distribution of I_t ? Specify parameters if needed.
2. Explain why there is an "inverse" process C s.t. $C_{A_t} = t$, $t < T$.
3. Define a stochastic process $B_t := I_{C_t}$. Prove that B_t has a centred normal distribution. Compute its covariance and deduce that B is a brownian motion[1]
4. Asume that $\lim_{\substack{t\to T\\<}} A_t = \infty$. Prove that $\limsup_{\substack{t\to T\\<}} I_t = \infty$ a.s.

The market offers a constant interest rate r and an asset modelled by

$$\frac{\mathrm{d}S_t}{S_t} \;=\; \mu \,\mathrm{d}t \;+\; \sigma \,\mathrm{d}W_t.$$

A trader starts a protfolio V with initial value \$1. He is allowed to trade in a self-financing manner, and he believes he has a guaranteed winning strategy that will earn him at least \$5 bn before time T. At each time t, he invests an amount $\$\frac{1}{\sqrt{T-t}}$ in the asset S.

5. Give the dynamic of $e^{-rt}V_t$, and prove that $e^{-rt}V_t = g(t) + I_t$ for a deterministic function g and an α to be determined explicitly.
6. Prove that this portfolio will obtain a value of at least \$5 bn a.s. before T.
7. Why might tis strategy fail in practice ?

Proof. 1. α is deterministic so

$$I_t \;\sim\; \mathcal{N}\left(0\,,\, \int_0^t |\alpha_s|^2 \,\mathrm{d}s\right).$$

2. With $C_t := \inf\{s : A_s \geq t\}$, we clearly have $C_{A_t} = t$.
3. We have $B_{A_t} = I_t \sim \mathcal{N}(0, A_t)$, hence $B_t \sim \mathcal{N}(0,t)$. And its covariance,

$$\begin{aligned} \mathbb{E}\left[B_{A_t} B_{A_s}\right] &= \mathbb{E}[I_t I_s] \\ &= \int_0^{t\wedge s} \alpha_u^2 \,\mathrm{d}u \\ &= A_t \wedge A_s. \end{aligned}$$

So $\mathbb{E}[B_t B_s] = t \wedge s$.

4.

$$\begin{aligned} \limsup_{\substack{t\to T\\<}} I_t &= \limsup_{\substack{t\to T\\<}} B_{A_t} \\ &= \limsup_{s\to\infty} B_s \;=\; \infty. \end{aligned}$$

5. We have the strategy $V_0 = 1$ and $\delta_t = \frac{1}{\sqrt{T-t}}$, so $\mathrm{d}V_t = \delta_t \frac{\mathrm{d}S_t}{S_t} + r(V_t - \delta_t)\,\mathrm{d}t$.

$$\begin{aligned} \mathrm{d}\left(e^{-rt}V_t\right) &= -re^{-rt}V_t \,\mathrm{d}t \;+\; e^{-rt}(\delta_t \mu \,\mathrm{d}t + \delta_t \sigma \,\mathrm{d}W_t + rV_t \,\mathrm{d}t - r\delta_t \,\mathrm{d}t) \\ &= \delta_t e^{-rt}(\mu - r)\,\mathrm{d}t \;+\; \delta_t e^{-rt}\sigma \,\mathrm{d}W_t \\ e^{-rt}V_t &= g(t) \;+\; I_t^\alpha\,, \end{aligned}$$

where $g(t) = 1 + \int_0^t \frac{\mu - r}{\sqrt{T-s}} e^{-rs}\,\mathrm{d}s$, $\alpha_t = \frac{\sigma}{\sqrt{T-t}} e^{-rt}$.

[1] We should actually prove that B is a gaussian vector to show that its a brownian motion.

6. We have $\lim_{t \underset{<}{\to} T} A_t = \lim_{t \underset{<}{\to} T} \int_0^t \frac{\sigma^2}{T-s} e^{-rs}\, \mathrm{d}s = \infty$, so with question 4,

$$\begin{aligned} & \limsup_{t \underset{<}{\to} T} I(t) = \infty \\ \Rightarrow \quad & \limsup_{t \underset{<}{\to} T} e^{-rt} V_t = \infty. \end{aligned}$$

7. The first problem is that $\delta \xrightarrow[t \to T]{} \infty$, and we can show that $\liminf_{t \underset{<}{\to} T} e^{-rt} V_t = -\infty$.

□

Exercise 3 (Clewlow–Strickland two factor model)**.** The market offers a constant rate of interest r. Let $W^{(1)}$ and $W^{(2)}$ be correlated Brownian motions with $\mathrm{d}\langle W^{(1)}, W^{(2)}\rangle = \rho\, \mathrm{d}t$ for $\rho \in]0,1[$ under the risk neutral probability measure $\mathbb{Q}$. For any finite $T > 0$ and $0 \le t \le T$, define the dynamical process

$$\frac{\mathrm{d}F_{t,T}}{F_{t,T}} = \sigma_1 e^{-\theta_1 (T-t)}\, \mathrm{d}W^{(1)} + \sigma_2 e^{-\theta_2 (T-t)}\, \mathrm{d}W^{(2)},$$

for $\sigma_1, \sigma_2, \theta_1, \theta_2 > 0$ and $F_0, T = 1$.

Show that the price at time t of a calendar spread option with payoff $(F_{T,T_1} - F_{T,T_2})^+$ for $T \le T_1 \le T_2$ is equal to

$$e^{-r(T-t} F_{t,T_1} \Phi(d_+) - e^{-r(T-t)} F_{t,T_2} \Phi(d_-), \qquad d_\pm = \frac{1}{\sqrt{v}} \ln\left(\frac{F_{t,T_1}}{F_{t,T_2}}\right) \pm \frac{1}{2}\sqrt{v},$$

$$\begin{aligned} v = \int_t^T & \left(\sigma_1^2 \left(e^{-\theta_1 (T_1 - s)} - e^{-\theta_1 (T_2 - s)} \right)^2 + \sigma_2^2 \left(e^{-\theta_2 (T_1 - s)} - e^{-\theta_2 (T_2 - s)} \right)^2 \right. \\ & \left. + 2\rho\sigma_1\sigma_2 \left(e^{-\theta_1 (T_1 - s)} - e^{-\theta_1 (T_2 - s)} \right) \left(e^{-\theta_2 (T_1 - s)} - e^{-\theta_2 (T_2 - s)} \right) \right) \mathrm{d}s. \end{aligned}$$

Proof. We define $B = (B^1, B^2)$ a brownian motion in $\mathbb{Q}$,

$$\begin{aligned} B^1 &= W^1; \\ B^2 &= \rho B^1 + \sqrt{1-\rho^2} B^2. \end{aligned}$$

So F_{t,T_1}, F_{t,T_2} are martingales under $\mathbb{Q}$ and with $i \in \{1,2\}$,

$$\begin{aligned} \frac{\mathrm{d}F_{t,T_i}}{F_{t,T_i}} &= \sigma_1 e^{-\theta_1 (T_i - t)}\, \mathrm{d}B_t^1 + \rho\sigma_2 e^{-\theta_2 (T_i - t)}\, \mathrm{d}B_t^1 + \sqrt{1-\rho^2}\sigma_2 e^{-\theta_2 (T_i - t)}\, \mathrm{d}B_t^2; \\ F_{t,T_2} &= \exp\left(\int_0^t \lambda_s^*\, \mathrm{d}B_s - \frac{1}{2} \int_0^t \|\lambda_s\|^2\, \mathrm{d}s \right), \end{aligned}$$

where

$$\lambda_S = \begin{bmatrix} \sigma_1 e^{-\theta_1 (T_2 - s)} + \rho\sigma_2 e^{-\theta_2 (T_2 - s)} \\ \sqrt{1-\rho^2}\sigma_2 e^{-\theta_2 (T_2 - s)} \end{bmatrix}.$$

We note P_t the price at time t,

$$\begin{aligned} P_t &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} F_{T,T_2} \left(\frac{F_{T,T_1}}{F_{T,T_2}} - 1 \right)^+ \middle| \mathcal{F}_t \right] \\ &= F_{T,T_2} \mathbb{E}_{\mathbb{Q}^{T_2}} \left[e^{-r(T-t)} \left(\frac{F_{T,T_1}}{F_{T,T_2}} - 1 \right)^+ \middle| \mathcal{F}_t \right] \qquad \text{(Bayes formula)} \end{aligned}$$

with

$$\frac{\mathrm{d}\mathbb{Q}^{T_2}}{\mathrm{d}\mathbb{Q}} = F_{T,T_2}.$$

And

$$
\begin{aligned}
F_{t,T_1} &= \exp\left(\int_0^t \vartheta_s^* \, \mathrm{d}B_s \;-\; \frac{1}{2}\int_0^t \|\vartheta_s\|^2 \, \mathrm{d}s\right), \qquad \text{with} \\
\vartheta_s &= \begin{bmatrix} \sigma_1 e^{-\theta_1(T_1-s)} \;+\; \rho\sigma_2 e^{-\theta_2(T_1-s)} \\ \\ \sqrt{1-\rho^2}\sigma_2 e^{-\theta_2(T_1-s)} \end{bmatrix}.
\end{aligned}
$$

So finally,

$$
\begin{aligned}
P_t &= e^{-r(T-t)} F_{t,T_2} \frac{F_{t,T_1}}{F_{t,T_2}} \Phi(d_+) \;-\; e^{-r(T-t)} F_{t,T_2} \Phi(d_-), \qquad \text{with} \\
d_\pm &= \frac{\ln \frac{F_{t,T_1}}{F_{t,T_2}}}{\sqrt{\int_t^T \|\lambda_s - \vartheta_s\|^2 \, \mathrm{d}s}} \;\pm\; \frac{1}{2}\int_t^T \|\lambda_s - \vartheta_s\|^2 \, \mathrm{d}s.
\end{aligned}
$$

□