Stochastic processes and derivatives
Sheet 6

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Exercise 1. Suppose that the rate of interest is random. Interest may be paid either in a continuous time or discrete time. Let $B(t,T)$ be the price of a zero-coupon bond with maturity T at time t. An important quantity is the forward rate, given by

$$\begin{aligned} f(t,T) &:= -\partial_T \ln B(t,T) && \text{in the continuous setting ;} \\ f(t_i,t_N) &:= \frac{B(t_i,t_N)}{B(t_i,t_{N+1})} - 1 && \text{in the discrete setting.} \end{aligned}$$

1. Prove that the forward rate can be used to express the price of the zero-coupon bond by

$$\begin{aligned} B(t,T) &= \exp\left(-\int_t^T f(t,s)\,\mathrm{d}s\right) && \text{in the continuous setting ;} \\ B(t_i,t_N) &= \prod_{j=1}^{N-1} \left(1+f(t_i,t_j)\right)^{-1} && \text{in the discrete setting.} \end{aligned}$$

We now investigate the long forward rate $f_L(t) := \lim_{T\to\infty} f(t,T)$.

2. The zero-coupon rate is defined by

$$\begin{aligned} z(t,T) &:= \frac{1}{T-t}\int_t^T f(t,s)\,\mathrm{d}s && \text{in the continuous setting ;} \\ z(t_i,t_N) &:= \frac{1}{B(t_i,t_N)^{\frac{1}{N-i}}} - 1 && \text{in the discrete setting.} \end{aligned}$$

   Prove that

$$\begin{aligned} \inf_{t\le s\le T} f(t,s) \;\le\; z(t,T) \;\le\; \sup_{t\le s\le T} f(t,s) && \text{in the continuous setting ;} \\ \inf_{t_i\le t_i\le t_N} f(t_i,t_j) \;\le\; z(t_i,t_N) \;\le\; \sup_{t_i\le t_j\le t_N} f(t_i,t_N) && \text{in the discrete setting.} \end{aligned}$$

   Show that, if the long forward rate exists almost surely, then the long zero-coupon rate $z_L(t) := \lim_{T\to\infty} z(t,T)$ exists almost surely and $z_L(t) = f_L(t)$.

3. Let $x_L(t) := \lim_{T\to\infty} B(t,T)^{\frac{1}{T}}$. Prove that $z_L(t)$ exists iff $x_L(t)$ exists.

4. Let X_n be a sequence of non-negative r.v. and let $\mathcal{G}$ be a sub-σ-algebra. Suppose that X_n converges a.s. to X, and $Y := \liminf_{n\to\infty} \mathbb{E}\left[X_n^n|\mathcal{G}\right]^{\frac{1}{n}} < \infty$ a.s. Prove that $\mathbb{E}[XZ] \le \mathbb{E}[YZ]$ for any non-negative, bounded r.v. Z, and prove that $X \le Y$ a.s.

5. Prove that, if $t \ge s \ge 0$ are such that $x_L(t)$ and $x_L(s)$ exist, then $x_L(s) \ge x_L(t)$.

6. Deduce that $f_L(t) \ge f_L(s)$, $\forall t \ge s \ge 0$.

Proof. 1. We have $B(t,t) = 1$ so the formula is obvious in the continuous case. For the discrete case let us proceed by recurrence,

$$B(t_i, t_i) = (1 + f(t_i, t_i))^{-1} = 1 \qquad \text{as } f(t,t) = 0.$$

Now we consider that the formula is true for t_{N-1},

$$\begin{aligned} B(t_i, t_N) &= B(t_i, t_{N-1})(1 + f(t_i, t_N))^{-1} \\ &= \prod_{j=1}^{N-2} (1 + f(t_i, t_j))^{-1}(1 + f(t_i, t_N))^{-1} \\ &= \prod_{j=1}^{N-1} (1 + f(t_i, t_j))^{-1} \end{aligned}$$

2. Again it is obvious in the continuous case. For the discrete setting,

$$\begin{aligned} & \inf_{t_i \le t_i \le t_N} f(t_i, t_j) \le f(t_i, t_j) \\ \Leftrightarrow \quad & \left(1 + \inf_{t_i \le t_i \le t_N} f(t_i, t_j)\right)^{-1} \le (1 + f(t_i, t_j))^{-1} \\ \Leftrightarrow \quad & \frac{1}{(1 + \inf_{t_i \le t_i \le t_N} f(t_i, t_j))^{N-i}} = \prod_{j=i}^{N-1} \left(1 + \inf_{t_i \le t_i \le t_N} f(t_i, t_j)\right)^{-1} \le \prod_{j=i}^{N-1} (1 + f(t_i, t_j))^{-1} = B(t_i, t_N) \\ \Leftrightarrow \quad & \inf_{t_i \le t_i \le t_N} f(t_i, t_j) \le \frac{1}{B(t_i, t_N)^{\frac{1}{N-i}}} - 1 = z(t_i, t_N). \end{aligned}$$

Now we assume that the long forward rate $f_L(t)$ exists, hence $\forall \varepsilon > 0$, $\exists T_\varepsilon$ s.t. $|f_L(t) - f(t,T)| < \varepsilon$, $\forall T \ge T_\varepsilon$. So

$$\begin{aligned} \underbrace{\int_0^{T_\varepsilon} f(t,s)\,\mathrm{d}s}_{=:\, I(T_\varepsilon)} + \int_{T_\varepsilon}^{T} (f_L(t) - \varepsilon)\,\mathrm{d}s &\le \int_t^T f(t,s)\,\mathrm{d}s \\ \underbrace{\frac{I(T_\varepsilon)}{T-t}}_{\xrightarrow[T\to\infty]{} 0} + \underbrace{\frac{T - T_\varepsilon}{T-t}}_{\xrightarrow[T\to\infty]{} 1} (f_L(t) - \varepsilon) &\le z(t,T), \end{aligned}$$

hence $f_L(t) - \varepsilon \le z(t,T) \le f_L(t) + \varepsilon$ and $z_L(t) = f_L(t)$.

3.

4.

$$\begin{aligned} \mathbb{E}[XZ] &= \mathbb{E}\left[\lim_n X_n Z\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\liminf_n X_n Z \middle| \mathcal{G}\right]\right] \\ &\le \mathbb{E}\left[\liminf_n \mathbb{E}\left[X_n Z | \mathcal{G}\right]\right] \qquad \text{(Fatou's Lemma)} \\ &\le \mathbb{E}\left[\liminf_n \mathbb{E}\left[X_n^n | \mathcal{G}\right]^{\frac{1}{n}} \lim_n \mathbb{E}\left[Z_n^{\frac{n-1}{n}} \middle| \mathcal{G}\right]^{\frac{n}{n-1}}\right] \qquad \text{(Hölder inequality)} \\ &\le \mathbb{E}[YZ]. \end{aligned}$$

Now we set $Z := \mathbb{1}_{\{X>Y\}}$, hence if $\mathbb{P}\{X > Y\} \ne 0$, we have $\mathbb{E}[(X-Y)Z] > 0$ as X is non-negative and $Y < X$, and we see the contradiction. So $X \le Y$ a.s.

5. We have, under the risk neutral measure, $B(t,T) = \mathbb{E}_{\mathbb{Q}}\left[\frac{R_t}{R_T}\middle|\mathcal{F}_t\right]$, where R_t is the accumulated interest from 0 to t. Under the change of measure $\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \frac{R_s}{B(s,t)R_t}$, $B(s,T) = B(s,T)\mathbb{E}_{\tilde{\mathbb{Q}}}[B(t,T)|\mathcal{F}_s]$. Hence,

$$\begin{aligned} x_L(s) &\triangleq \lim_{T\to\infty} B(s,T)^{\frac{1}{T}} \\ &= \lim_{T\to\infty} B(s,t)^{\frac{1}{T}}\mathbb{E}_{\tilde{\mathbb{Q}}}[B(t,T)|\mathcal{F}_s]^{\frac{1}{T}} \\ &= \lim_{T\to\infty} B(s,t)^{\frac{1}{T}}\mathbb{E}_{\tilde{\mathbb{Q}}}[x(t,T)^T|\mathcal{F}_s]^{\frac{1}{T}} \qquad \text{by setting } x(t,T) := B(t,T)^{\frac{1}{T}} \xrightarrow[T\to\infty]{} x_L(t) \\ &\geq \underbrace{\lim_{T\to\infty} B(s,t)^{\frac{1}{T}}}_{=0}\ \underbrace{\liminf_n \mathbb{E}_{\tilde{\mathbb{Q}}}[x(t,T)^T|\mathcal{F}_s]^{\frac{1}{T}}}_{\geq x_L(t)} \\ &\geq x_L(t). \end{aligned}$$

6. Now as $z_L(t) = f_L(t) = -\ln x_L(t)$, f_L is a growing function with time.

□

Exercise 2 (Zero coupon bond in Vasicek model)**.** Consider the interest rate model of Vasicek under the risk-neutral measure $\mathbb{Q}$:

$$\mathrm{d}r_t \;=\; a(b-r_t)\,\mathrm{d}t \;-\; \sigma\,\mathrm{d}W_t\,, \qquad r_0 \;=\; x.$$

Part 1: pricing of the zero-coupon bond.

1. Prove that

$$r_t \;=\; r_0e^{-at} \;+\; b(1-e^{-at}) \;-\; \sigma\int_0^t e^{-a(t-s)}\,\mathrm{d}W_s\,,$$

and deduce that r is a Gaussian process with explicit mean and covariance parameters.

2. We now define $I_{t,T} := \int_t^T r_s\,\mathrm{d}s$. Prove that

$$aI_{t,T} \;=\; ab(T-t) \;-\; (r_T-r_t) \;-\; \sigma\int_t^T \mathrm{d}W_s\,,$$

and deduce the explicit form of $I_{t,T}$ with respect to r_t.

3. Compute the conditional mean and variance of $I_{t,T}$ and deduce the price of the zero-coupon bond $B(t,T)$,

$$B(t,T) \;=\; \exp\left(-b(T-t)+(b-r_t)\frac{1-e^{-a(T-t)}}{a} - \frac{\sigma^2}{4a^3}\left(1-e^{-a(T-t)}\right)^2 + \frac{\sigma^2}{2a^2}\left(T-t-\frac{1-e^{-a(T-t)}}{a}\right)\right).$$

Part 2: pricing of a call option on a forward contract. Let now $\Gamma(t,T)$ be the volatility of the zero-coupon bond. The forward neutral with maturity T probability $\mathbb{Q}^T$ is the measure with Radon-Nikodym derivative $\exp\left(-\int_0^T r_s\,\mathrm{d}s\right)B(0,T)^{-1}$.

4. What is the value of $\Gamma(t,T)$? What is the SDE of the ZC bond under the risk neutral probability, and what is it under the forward-neutral with maturity T probability ? Find the solution to these SDEs.

5. For any $\delta \geq 0$, show that the forward price at time t for the payoff $B(T,T+\delta)^{-1}$ at maturity T is $X_t := \mathbb{E}_{\mathbb{Q}^T}[B(T,T+\delta)^{-1}|\mathcal{F}_t]$ and compute it.

Proof. 1. Let $X_t = r_0e^{-at} + b(1-e^{-at}) - \sigma\int_0^t e^{-a(t-s)}\,\mathrm{d}W_s$, then with Itô's lemma,

$$\begin{aligned} \mathrm{d}X_t &= \left(-ar_0e^{-at}+abe^{-at}\right)\,\mathrm{d}t \;-\; \sigma\,\mathrm{d}\left(\int_0^t e^{-a(t-s)}\,\mathrm{d}W_s\right) \\ &= \left(-ar_0e^{-at}+abe^{-at}+ab-ab\right)\,\mathrm{d}t \;-\; \sigma\left(e^{-a(t-t)}\,\mathrm{d}W_t + a\left(\int_0^t e^{-a(t-s)}\,\mathrm{d}W_s\right)\,\mathrm{d}t\right) \\ &= a(b-X_t)\,\mathrm{d}t \;-\; \sigma\,\mathrm{d}W_t. \end{aligned}$$

So the right hand side of r_t is a Wiener integral, then it is a Gaussian process with mean and covariance:

$$\begin{aligned}\mathbb{E}[r_t] &= r_0 e^{-at} + b(1-e^{-at}) \\ \mathrm{Cov}(r_t, r_s) &\triangleq \sigma^2 \int_0^{t\wedge s} e^{-a(t-u)} e^{-a(s-u)}\,\mathrm{d}u \\ &= \frac{\sigma^2}{2}\left(e^{-a|t-s|} - e^{-a(t+s)}\right).\end{aligned}$$

2. We have

$$\begin{aligned} & \mathrm{d}r_s = a(b-r_s)\,\mathrm{d}s - \sigma\,\mathrm{d}W_s \\ \Leftrightarrow\quad & \int_t^T \mathrm{d}r_s = \int_t^T a(b-r_s)\,\mathrm{d}s - \sigma\int_t^T \mathrm{d}W_s \\ \Leftrightarrow\quad & r_T - r_t = ab(T-t) - aI_{t,T} - \sigma\int_t^T \mathrm{d}W_s.\end{aligned}$$

We also know from 1 that

$$r_T = r_t e^{-a(T-t)} + b(1-e^{-a(T-t)}) - \sigma\int_t^T e^{-a(T-s)}\,\mathrm{d}W_s$$

hence,

$$I_{t,T} = b(T-t) + (r_t - b)\frac{1-e^{-a(T-t)}}{a} + \frac{\sigma}{a}\int_t^T \left(e^{-a(T-s)} - 1\right)\mathrm{d}W_s$$

3. We note $m^t_{I_{t,T}}$ and $s^t_{I_{t,T}}$ the mean and variance of $I_{t,T}$ conditionally to $\mathcal{F}_t$,

$$\begin{aligned} m^t_{I_{t,T}} &= b(T-t) + (r_t-b)\frac{1-e^{-a(T-t)}}{a} \\ s^t_{I_{t,T}} &\triangleq \frac{\sigma^2}{a^2}\int_t^T \left(e^{-a(T-s)}-1\right)^2 \mathrm{d}s \\ &= \frac{\sigma^2}{a^2}\left[\frac{1}{2a}e^{-2a(T-s)} + s - \frac{2}{a}e^{-a(T-s)}\right]_t^T \\ &= \frac{\sigma^2}{a^2}\left(T-t-\frac{1-e^{-a(T-t)}}{a} - \left(1-e^{-a(T-t)}\right)^2\right).\end{aligned}$$

Now we deduce the price of a zero-coupon bond,

$$\begin{aligned} B(t,T) &\triangleq \mathbb{E}\left[e^{-I_{t,T}}\middle|\,\mathcal{F}_t\right] \\ &= e^{-m^t_{I_{t,T}} + \frac{1}{2}s^t_{I_{t,T}}}.\end{aligned}$$

4. We have with Itô's Lemma,

$$\mathrm{d}B(t,T) = \cdot\,\mathrm{d}t + \sigma\frac{1-e^{-a(T-t)}}{a}\,\mathrm{d}W_t\,,$$

hence by identification, $\Gamma(t,T) = \sigma\frac{1-e^{-a(T-t)}}{a}$.
Under the risk neutral probability,

$$\frac{\mathrm{d}B(t,T)}{B(t,T)} = r_t\,\mathrm{d}t + \Gamma(t,T)\,\mathrm{d}W_t.$$

And the solution is

$$B(t,T) = B(0,T)\exp\left(\int_t^T r_s - \frac{1}{2}\Gamma(s,T)^2\,\mathrm{d}s + \int_t^T \Gamma(s,T)\,\mathrm{d}W_s\right).$$

We have $B(T,T)=1$, hence $\frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = \mathcal{E}\left(\int_t^T \Gamma(s,T)\,\mathrm{d}W_s\right)$. With Girsanov's theorem, $\mathrm{d}W_t^{\mathbb{Q}^T} := \mathrm{d}W_t - \Gamma(t,T)\,\mathrm{d}t$ brownian motion under $\mathbb{Q}^T$, and then under $\mathbb{Q}^T$,

$$\begin{aligned} \frac{\mathrm{d}B(t,T)}{B(t,T)} &= (r_t + \Gamma(t,T))\,\mathrm{d}t \;+\; \Gamma(t,T)\,\mathrm{d}W_t^{\mathbb{Q}^T} \\ \Leftrightarrow \quad B(t,T) &= B(0,T)\exp\left(\int_t^T r_s + \frac{1}{2}\Gamma(s,T)^2\,\mathrm{d}s \;+\; \int_t^T \Gamma(s,T)\,\mathrm{d}W_s^{\mathbb{Q}^T}\right). \end{aligned}$$

5. We have,

$$\begin{aligned} X_t &\triangleq B(t,T)^{-1}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_s\,\mathrm{d}s} B(T,T+\delta)^{-1} \middle| \mathcal{F}_t \right] \\ &= B(t,T)^{-1}\mathbb{E}_{\mathbb{Q}^T}\left[\frac{B(t,T)}{e^{-\int_t^T r_s\,\mathrm{d}s}} e^{-\int_t^T r_s\,\mathrm{d}s} B(T,T+\delta)^{-1} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}^T}\left[B(T,T+\delta)^{-1} \middle| \mathcal{F}_t \right]. \end{aligned}$$

6.

□