Stochastic processes and derivatives
Sheet 5

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Exercise 1. The dynamic of the price of a dividend paying asset satisfies the SDE

$$\frac{\mathrm{d}S_t}{S_t} \;=\; (r-q)\,\mathrm{d}t \;+\; \sigma\,\mathrm{d}W_t\,,$$

where W is a brownian motion under the risk neutral measure $\mathbb{Q}$, $\sigma > 0$, $r \geq 0$ rate of interest and q dividend rate. In this exercise we will prove the Call-Put symmetry

$$Call(T, S_0 e^{-\nu T}, K) \;=\; Put(T, K e^{-\nu T}, S_0)\,, \qquad \nu := r - q.$$

1. For $S_0 = x$, show that the price of $Call(T, x, k)$ under the numeraire

   $$X_t \;=\; \frac{S_t e^{-\nu t}}{x}$$

   is equal to

   $$\mathbb{E}_{\mathbb{Q}^X}\left[e^{-qT}\left(x - \frac{kx}{S_T}\right)^+\right].$$

2. Let $W_t^X := W_t - \sigma t$. Write down the formula for $\frac{kx}{S_T}$ as an exponential of W_T^X.
3. Using the fact that W_T^X is a brownian motion under $\mathbb{Q}^X$, prove the Call-Put symmetry.

Proof. 1.

$$\begin{aligned} Call(T,x,k) &\triangleq \mathbb{E}_{\mathbb{Q}}[e^{-rT}(S_T - k)^+] \\ &= \mathbb{E}_{\mathbb{Q}^X}\left[\frac{e^{-rT}}{X_T}(S_T - k)^+\right] \\ &= \mathbb{E}_{\mathbb{Q}^X}\left[e^{-qT}\left(x - \frac{kx}{S_T}\right)^+\right]. \end{aligned}$$

2. k is a martingale hence,

$$\begin{aligned} & \frac{\mathrm{d}\left(\frac{k}{X_T}\right)}{\frac{k}{X_T}} = (0-\sigma)\,\mathrm{d}W_T^X \\ \Leftrightarrow\quad & \mathrm{d}\left(\frac{kx}{S_T}e^{\nu T}\right) = -\frac{kx}{S_T}e^{\nu T}\sigma\,\mathrm{d}W_T^X = -ke^{\frac{\sigma^2}{2}T}\sigma\,\mathrm{d}W_T^X \\ \Leftrightarrow\quad & \frac{kx}{S_T}e^{\nu T} = ke^{-\sigma W_T^X - \frac{\sigma^2}{2}T} \\ \Leftrightarrow\quad & \frac{kx}{S_T} = ke^{-\nu T}e^{-\sigma W_T^X - \frac{\sigma^2}{2}T}. \end{aligned}$$

3.

$$\begin{aligned}
Call(T, S_0e^{-\nu T}, K) &= \mathbb{E}_{\mathbb{Q}^X}\left[e^{-qT}\left(S_0e^{-\nu T} - \frac{kS_0}{S_T}\right)^+\right] \\
&= \mathbb{E}_{\mathbb{Q}^X}\left[e^{-qT}\left(S_0e^{-\nu T} - Ke^{-\nu T}e^{-\sigma W_T^X - \frac{\sigma^2}{2}T}\right)^+\right]. \\
Put(T, x, K) &= \mathbb{E}_{\mathbb{Q}}[e^{-rT}(K - S_T)^+] \\
&= \mathbb{E}_{\mathbb{Q}}[e^{-qT}(e^{-\nu T}K - xe^{\sigma W_T - \frac{\sigma^2}{2}T})^+]. \\
Put(T, Ke^{-\nu T}, S_0) &= \mathbb{E}_{\mathbb{Q}^X}\left[e^{-qT}\left(S_0e^{-\nu T} - Ke^{-\nu T}e^{\sigma W_T - \frac{\sigma^2}{2}T}\right)^+\right].
\end{aligned}$$

The equality is proven as $\mathcal{L}_{\mathbb{Q}}(W) = \mathcal{L}_{\mathbb{Q}}(-W) = \mathcal{L}_{\mathbb{Q}^X}(W^X)$.

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Exercise 2. Assume that the market offers a constant rate of interest r and that the price of an asset is given by

$$\frac{\mathrm{d}S_t}{S_t} = r\,\mathrm{d}t + \sigma\,\mathrm{d}W_t.$$

We compute the price of the Asian option with payoff $\Psi_T := \left(\exp\left(\frac{1}{T}\int_0^T \ln S_t\,\mathrm{d}t\right) - KS_T\right)^+$.

1. Prove that $\frac{1}{T}\int_0^T \ln S_t\,\mathrm{d}t$ has a Gaussian distribution. Compute the mean and variance of this distribution.
2. Show that the price of the Asian option at time 0 can be expressed as

$$\mathbb{E}[(C_1\exp(Z + C_2T) - C_3K)^+]$$

for a gaussian r.v. Z and C_1, C_2, C_3 constants. The parameters of Z and the constants are to be given explicitly.

Proof. 1. We have with the definition of the integral,

$$\begin{aligned}
\int_0^T \ln S_t\,\mathrm{d}t &= \lim_{N\to\infty}\sum_{i=0}^{N-1}\ln S_{t_i}(t_{i+1} - t_i)\,, \qquad t_i = \frac{iT}{N} \\
&= \lim_N \sum_i \left(\ln S_0 + \sigma W_{t_i} + \left(r - \frac{\sigma^2}{2}\right)t_i\right)(t_{i+1} - t_i) \\
&= T\ln S_0 + \sigma\lim_N\sum_i W_{t_i}(t_{i+1} - t_i) + \left(r - \frac{\sigma^2}{2}\right)\int_0^T t\,\mathrm{d}t \\
&= T\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)\frac{T^2}{2} + \lim\Sigma_N^\sigma.
\end{aligned}$$

Where,

$$\begin{aligned}
\Sigma_N^\sigma &:= \sum_i \sigma W_{t_i}(t_{i+1} - t_i) \\
&= \sum_i (W_{t_{i+1}} - W_{t_i})\beta_i^\sigma \qquad \text{where } \beta_i^\sigma := \sigma\sum_{j=i}^{N-1}(t_{j+1} - t_j) = (T - t_j) \\
&\sim \mathcal{N}\left(0, \sigma^2\sum_i (T - t_i)^2(t_{i+1} - t_i)\right).
\end{aligned}$$

Hence

$$\Sigma_N^\sigma \xrightarrow[N\to\infty]{} \mathcal{N}(0, \underbrace{\sigma^2\int_0^T (T - t)^2\,\mathrm{d}t)}_{\sigma^2\frac{T^3}{3}}$$

And finally $\frac{1}{T}\int_0^T \ln S_t\,\mathrm{d}t \sim \mathcal{N}(\ln S_0 + (r - \frac{\sigma^2}{2})\frac{T}{2}, \sigma^2\frac{T^3}{3})$.

2.

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Exercise 3. At time t, the market offers a bounded interest rate r_t and two assets whose prices are given by

$$\begin{aligned}\frac{\mathrm{d}S_t^{(1)}}{S_t^{(1)}} &= b_t^{(1)}\,\mathrm{d}t \;+\; \sigma_1\,\mathrm{d}B_t^{(1)} \\ \frac{\mathrm{d}S_t^{(2)}}{S_t^{(2)}} &= b_t^{(2)}\,\mathrm{d}t \;+\; \sigma_2\left(\rho\,\mathrm{d}B_t^{(1)} + \sqrt{1-\rho^2}\,\mathrm{d}B_t^{(2)}\right),\end{aligned}$$

where $\left(B^{(1)}, B^{(2)}\right)$ is a two dimensional standard brownian motion under the historical measure $\mathbb{P}$, $\rho \in [-1,1]$, $\sigma_1, \sigma_2 > 0$, and $\left(b^{(1)}, b^{(2)}\right)$ is a bounded $\mathbb{R}^2$-valued process.

1. Compute the value of an option with payoff $\Psi = \left(S_T^{(2)} - KS_T^{(1)}\right)^+$ at time 0 for $K \geq 0$.
2. Give the hedging strategy invested in the risky assets only for the option.

Proof. 1. We have under the risk neutral measure $\mathbb{Q}$

$$\begin{aligned}\frac{\mathrm{d}S_t^{(1)}}{S_t^{(1)}} &= r_t\,\mathrm{d}t \;+\; \Sigma_1^*\cdot\mathrm{d}B_t^{\mathbb{Q}}, \qquad \Sigma_1 := \begin{pmatrix}\sigma_1\\ 0\end{pmatrix} \\ \frac{\mathrm{d}S_t^{(2)}}{S_t^{(2)}} &= r_t\,\mathrm{d}t \;+\; \Sigma_2^*\cdot\mathrm{d}B_t^{\mathbb{Q}}, \qquad \Sigma_2 := \begin{pmatrix}\sigma_2\rho\\ \sigma_2\sqrt{1-\rho^2}\end{pmatrix}\end{aligned}$$

where $B_t^{\mathbb{Q}}$ is a 2-dimensional brownian motion. We also define the change of numeraire

$$\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}}_{|\mathcal{F}_t} = \frac{e^{-\int_0^T r_s\,\mathrm{d}s}S_t^{(1)}}{S_0^{(1)}},$$

hence the price of the option is

$$\begin{aligned}C_0 &\triangleq \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r_s\,\mathrm{d}s}\left(S_T^{(2)} - KS_T^{(1)}\right)^+\right] \\ &= \mathbb{E}_{\mathbb{Q}^S}\left[S_0^{(1)}\left(\frac{S_T^{(2)}}{S_T^{(1)}} - K\right)^+\right]\end{aligned}$$

and $\frac{S_T^{(2)}}{S_T^{(1)}}$ is a martingale under $\mathbb{Q}^S$,

$$\begin{aligned}\frac{\mathrm{d}\frac{S_T^{(2)}}{S_T^{(1)}}}{\frac{S_T^{(2)}}{S_T^{(1)}}} &= (\Sigma_2 - \Sigma_1)\cdot\mathrm{d}B_t^S \\ &= \|\Sigma\|\,\mathrm{d}W_t^S,\end{aligned}$$

where B^S is a brownian motion under $\mathbb{Q}^2$, $\Sigma := \Sigma_2 - \Sigma_1$, $\mathrm{d}W_t^S := \|\Sigma\|^{-1}\left((\sigma_2\rho - \sigma_1)\,\mathrm{d}B_t^{S,1} + \sigma_2\sqrt{1-\rho^2}\,\mathrm{d}B_t^{S,2}\right)$. Hence we have a BS price:

$$\begin{aligned}\frac{C_0}{S_0^{(1)}} &= \frac{S_0^{(2)}}{S_0^{(1)}}\Phi(d^+) \;-\; K\Phi(d^-)\,; \\ d^{\pm} &:= \frac{\ln\frac{S_0^{(2)}}{S_0^{(1)}K}}{\|\Sigma\|\sqrt{T}} \pm \frac{1}{2}\|\Sigma\|\sqrt{T}\end{aligned}$$

2. Then the hedging strategy is to buy $\Phi(d^+)$ asset 2 and short $K\Phi(d^-)$ asset 1.

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Exercise 4. Assume that the market offers interest rate $r = 0$ and an asset whose price is $\frac{\mathrm{d}S_t}{S_t} = \sigma\,\mathrm{d}W_t$ for some bounded, possibly random $\sigma_t > 0$. In this question, we prove the Lee formula for the small strike asymptotic:

$$\limsup_{x\to-\infty} \frac{\sigma_I^2(x)}{|x|} \;=\; \beta_L \;:=\; 2 - 4\left(\sqrt{\tilde{q}^2+\tilde{q}}-\tilde{q}\right), \qquad \tilde{q} \;:=\; \sup\left\{q : \mathbb{E}\left[S_T^{-q}\right] < \infty\right\}$$

where $x := \ln\frac{K}{S_0}$ is the log-moneyness and $\sigma_I(x)$ is the BS implied volatility. You are given the formula for the large strike asymptotic:

$$\limsup_{x\to\infty} \frac{\sigma_I^2(x)}{|x|} \;=\; \beta_R \;:=\; 2 - 4\left(\sqrt{\tilde{p}^2+\tilde{p}}-\tilde{p}\right), \qquad \tilde{p} \;:=\; \sup\left\{p : \mathbb{E}\left[S_T^{p+1}\right] < \infty\right\}.$$

1. Notice that

$$\mathbb{E}\left[(S_t-K)^+\right] \;=\; C^{BS(\sigma_i(x))}(S_0,K),$$

where $C^{BS(\sigma)}(\cdot)$ is the BS (volatility σ) price of a call. Apply a change of numeraire to show that

$$\mathbb{E}\left[(S_t-K)^+\right] \;=\; P^{BS(\tilde{\sigma}_i(x))}(K,S_0),$$

where $P^{BS(\sigma)}(\cdot)$ is the BS (volatility σ) price of a put, and $\tilde{\sigma}_I(x)$ is the implied volatility of the fictional asset whose price is $\tilde{S}_t = \frac{S_0K}{S_t}$.

2. Use the put-call symmetry to show that $\sigma_I(x) = \tilde{\sigma}_I(-x)$ and therefore

$$\limsup_{x\to-\infty} \frac{\sigma_I^2(x)}{|x|} \;=\; \limsup_{x\to\infty} \frac{\tilde{\sigma}_I^2(x)}{x}.$$

3. Use the large strike asymptotic Lee formula for the asset $\tilde{S}$ in order to prove the small strike asymptotic Lee formula for the asset S.

Proof. 1. We have by definition of the implied volatility

$$\mathbb{E}\left[(S_t-K)^+\right] \;\triangleq\; C^{BS(\sigma_i(x))}(S_0,K).$$

Now we want to apply a change of numeraire

$$\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}}_{|\mathcal{F}_t} \;=\; \frac{S_t}{S_0},$$

hence the price is

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[(S_T-K)^+\right] &= \mathbb{E}_{\mathbb{Q}^S}\left[\frac{S_0}{S_T}(S_T-K)^+\right] \\
&= \mathbb{E}_{\mathbb{Q}^S}\left[\left(S_0 - \frac{KS_0}{S_T}\right)^+\right], \qquad \tilde{S}_T \;=\; \frac{KS_0}{S_T} \\
&= P^{BS(\tilde{\sigma}_I(-x))}(K,S_0).
\end{aligned}$$

Indeed as the spot is K and the strike S_0, we have $\tilde{\sigma}_I\left(\ln\frac{S_0}{K}\right) = \tilde{\sigma}_I(-x)$.

2. The symmetry (problem 1) writes $P^{BS(\sigma)}(K,S_0) = C^{BS(\sigma)}(S_0,K)$, hence in our case

$$C^{BS(\sigma_I(x))}(S_0,K) \;=\; C^{BS(\tilde{\sigma}_I(-x))}(S_0,K).$$

As the BS price is strictly growing with the volatility we have $\sigma_I(x) = \tilde{\sigma}_I(-x)$. And then

$$\begin{aligned}
\limsup_{x\to-\infty} \frac{\sigma_I^2(x)}{|x|} &= \limsup_{x\to-\infty} \frac{\tilde{\sigma}_I^2(-x)}{|x|} \\
&= \limsup_{y\to\infty} \frac{\tilde{\sigma}_I^2(y)}{|-y|}, \qquad y := -x \\
&= \limsup_{y\to\infty} \frac{\tilde{\sigma}_I^2(y)}{y}, \qquad \text{as } |-y| = |y| = y \text{ when } y\to\infty.
\end{aligned}$$

3. The large strike asymptotic Lee formula on $\tilde{S}$ is

$$\begin{aligned}
\limsup_{x\to\infty} \frac{\tilde{\sigma}_I^2(x)}{|x|} &= \beta_R := 2 - 4\left(\sqrt{\tilde{p}^2 + \tilde{p}} - \tilde{p}\right), \qquad \tilde{p} := \sup\left\{p : \mathbb{E}_{\mathbb{Q}}\left[\tilde{S}_T^{p+1}\right] < \infty\right\} \\
&= \limsup_{x\to-\infty} \frac{\sigma_I^2(x)}{|x|}.
\end{aligned}$$

And we have with a change of numeraire

$$\begin{aligned}
\tilde{p} &= \sup\left\{p : \mathbb{E}_{\mathbb{P}}\left[\frac{S_T}{S_0}\left(\frac{S_0 K}{S_T}\right)^{p+1}\right] < \infty\right\} \\
&= \sup\left\{p : \mathbb{E}_{\mathbb{P}}\left[S_T^{-p}\right] < \infty\right\} \\
&= \tilde{q}.
\end{aligned}$$

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