Stochastic processes and derivatives
Sheet 4

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Exercise 1 (Delta–Gamma hedging)**.** Suppose that the market offers a fixed interest rate r and the price $(S_t)_{t\geq 0}$ of an asset follows the model

$$\frac{\mathrm{d}S_t}{S_t} \;=\; r\,\mathrm{d}t \;+\; \sigma\,\mathrm{d}W_t\,, \qquad S_0 \;=\; x.$$

Consider a european option on the asset whose price at time t is given by $u(t,S_t)$, where u is a smooth function with bounded derivatives. the option maturity is at time T. We consider discrete time trading strategies: trading only takes place at N-time points in the interval $[0,T]$ at the times $\{0=t_1<\cdots<t_N<T\}$ with $|t_i-t_{i+1}|\leq \frac{T}{N}$.

1. The value of the hedging portfolio containing only assets and bonds is V_t. Prove that
$$\mathrm{d}(e^{-rt}V_t^N) \;=\; u_x(t,S_t)\,\mathrm{d}(e^{-rt}S_t)$$
2. The self-financing portfolio of the discrete time Delta hedging strategy is
$$\mathrm{d}(e^{-rt}V_t^N) \;=\; u_x(t_i,S_{t_i})\,\mathrm{d}(e^{-rt}S_t)$$
for $t\in[t_i,t_{i+1}[$, $i\in[\![0,N-1]\!]$. Prove that the expected square tracking error (*i.e.* $\mathbb{E}[|V_T-V_T^N|^2]$) of the discrete time Delta hedging strategy is of order $O(N^{-1})$.

Proof. 1.

$$\begin{aligned}
\mathrm{d}(e^{-rt}V_t) &\triangleq \mathrm{d}(e^{-rt}u(t,S_t))\\
&= u(t,S_t)\,\mathrm{d}(e^{-rt}) \;+\; e^{-rt}\,\mathrm{d}(u(t,S_t)) \;+\; \mathrm{d}\langle e^{-r\cdot},u(\cdot,S_\cdot)\rangle_t\\
&= -re^{-rt}u(t,S_t)\,\mathrm{d}t \;+\; e^{-rt}\left((u_t+\frac{1}{2}u_{xx}\sigma^2S_t^2+u_xrS_t)\,\mathrm{d}t+u_x\sigma S_t\,\mathrm{d}W_t\right)\\
&= e^{-rt}\underbrace{(-ru+u_t+\frac{1}{2}u_xx\sigma^2S_t^2+rS_tu_x)}_{=0\text{ with the BS equation}}\mathrm{d}t \;+\; u_x\sigma e^{-rt}S_t\,\mathrm{d}W_t.\\
&= u_x(t,S_t)\,\mathrm{d}(e^{-rt}S_t).
\end{aligned}$$

2. We have $V_0^N=V_0$, and with the last question we got the dynamic of $e^{-rt}V_t$ which leads to
$$e^{-rT}V_T \;=\; V_0 \;+\; \int_0^T u_x(t,S_t)\,\mathrm{d}\tilde{S}_t\,,$$
where $\tilde{S}_t:=e^{-rt}S_t$. We also have the dynamic of $e^{-rt}V_t^N$, *i.e.* $\mathrm{V}_t^N=u_x(\varphi(t),S_{\varphi(t)})\,\mathrm{d}\tilde{S}_t$, with $\varphi(t):=\sup_i\{t_i: t_i\leq t\}$. And so,
$$e^{-rT}V_T^N \;=\; V_0 \;+\; \int_0^T u_x(\varphi(t),S_{\varphi(t)})\,\mathrm{d}\tilde{S}_t.$$
Recall that $\mathrm{d}\tilde{S}_t=e^{-rt}\sigma S_t\,\mathrm{d}W_t$, hence
$$\begin{aligned}
\mathbb{E}[|V_T-V_T^N|^2] &= e^{2rT}\mathbb{E}\left[\left|\int_0^T (u_x(t,S_t)-u_x(\varphi(t),S_{\varphi(t)}))e^{-rt}\sigma\,\mathrm{d}W_s\right|\right]\\
&= e^{2rT}\sigma^2\mathbb{E}\left[\int_0^T |M_t|^2\,\mathrm{d}t\right]\,,
\end{aligned}$$

where $M_t := D_u(t)e^{-rt}S_t$ and $D_u(t) := u_x(t, S_t) - u_x(\varphi(t), S_{\varphi(t)})$.

□

Exercise 2. The market offers interest rate zero and an asset whose price is given by $xe^{-\sigma^2 \frac{t}{2} + \sigma W_t}$, where W is a Brownian motion. The asset offers a dividend zero. We consider the barrier option which offers the holder payoff $(S_{T+\theta} - K)^+$ at time $T + \theta$ if the stock proce descends below a barrier $H \in]0, x \vee K[$ before time T.

1. Prove that

$$\frac{K}{H} Put(t, x, T, \frac{H^2}{K}) = \quad Call(t, H, T, \tfrac{Kx}{H}).$$

   Compute the value of the barrier option if $\theta = 0$.

2. Recall that, from the pricing formula of the binary DIC option, we have that

$$\mathbb{P}\left\{\inf_{t\in[0,T]} S_t \leq H,\, S_T > K\right\} \quad = \quad \frac{x}{H}\mathbb{P}\left\{S_T > \frac{Kx^2}{H^2}\right\}$$

   for all $K \geq H$ and $x \geq H$.

Proof. 1.

$$\begin{aligned}
\frac{K}{H} Put(t, x, T, \frac{H^2}{K}) &= \frac{K}{H}\mathbb{E}\left[\left(\frac{H^2}{K} - xe^{\sigma W_{T-t} - \frac{1}{2}\sigma^2(T-t)}\right)^+\right] \\
&= \mathbb{E}\left[\left(H - \frac{Kx}{H}e^{\sigma W_{T-t} - \frac{1}{2}\sigma^2(T-t)}\right)^+\right] \\
&= Call(t, H, T, \frac{Kx}{H}).
\end{aligned}$$

For a regular DIC, if $x \leq H$, we have a call and then,

$$DIC(t, x, T, K, H) \quad = \quad Call(t, x, T, K).$$

If $x > H$, we note $T_H := \inf\{s \geq t : S_s \leq H\}$. Then at $t = T_H$,

$$\begin{aligned}
DIC(T_H, H, T, K, H) &= Call(T_H, H, T, K) \\
&= \frac{K}{H} Put(T_H, H, T, \frac{H^2}{K}).
\end{aligned}$$

The option whose price at the barrier is $Put(T_H, H, T, \frac{H^2}{K})$ is a Down and In Put, hence $DIC(t, x, T, K, H) = \frac{K}{H} DIP(t, x, T, \frac{H^2}{K}, H)$. The put is exercised only if the underlying price at T is less than $\frac{H^2}{K} < H$, hence the barrier has been reached. So its value is just $Put(t, x, T, \frac{H^2}{K})$, and we conclude with the symmetry,

$$DIC(t, x, T, K, H) \quad = \quad Call(t, H, T, \frac{Kx}{H}).$$

2.

$$\begin{aligned}
\mathbb{P}\left\{\inf_{t\in[0,T]} S_t \leq H,\, S_T > K\right\} &= \mathbb{E}\left[\mathbb{1}_{\{T_H < T\}}\mathbb{1}_{\{S_T > K\}}\right] \\
&= BinDIC(0, x, T, K, H) \\
&= -\partial_k DIC(0, x, T, k, H)|_{k=K} \\
&= -\frac{x}{H}\partial_k Call(0, H, T, k)|_{k=\frac{Kx}{H}} \\
&= \frac{x}{H} BinCall(0, H, T, \frac{Kx}{H}) \\
&= \frac{x}{H}\mathbb{E}\left[\mathbb{1}_{\{He^{\sigma W_T - \frac{\sigma^2}{2}T} > \frac{Kx}{H}\}}\right] \\
&= \frac{x}{H}\mathbb{P}\left\{S_T > \frac{Kx^2}{H^2}\right\}.
\end{aligned}$$

□

Exercise 3. The investor wants to replicate a "power" asset which is not available in the market. According to the value of the power exponent, the investor is risk-averse or risk-taker. We derive a hedging strategy for replicating the payoff at maturity.

The market offers a constant interest rate $r \geq 0$, and contains an asset whose price is given by

$$\frac{\mathrm{d}S_t}{S_t} = r\,\mathrm{d}t + \sigma(\mathrm{d}W_t + \lambda\,\mathrm{d}t),$$

where W is a Brownian motion, $\lambda \in \mathbb{R}$ and $\sigma > 0$. One wishes to invest in a self-financing portfolio whose value at time T is

$$V_T := y(S_T)^\gamma, \qquad \gamma > 0,\ y > 0.$$

1. Show that the price of the portfolio at time t is satisfied by $V_t = u(t, S_t)$, where $u(t,x) = yA(t,T)x^\gamma$ for a function $A(\cdot,T) : [0,T] \to \mathbb{R}$ to be found explicitly. Determine the composition of the self-financing portfolio, *i.e.* find the quantity of assets and the quantity of bonds held at each time.

2. Write down the stochastic differential equation satisfied by $X_t = yS_t^\gamma$ under the risk neutral measure. Verify that $V_t = A(t,T)X_t$ and that

$$\frac{\mathrm{d}V_t}{V_t} = r\,\mathrm{d}t + \sigma\gamma\,\mathrm{d}\tilde{W}_t,$$

where $\tilde{W}_t := W_t + \lambda t$.

3. The holder wants to insure the portfolio V_t by buying a put option written on it with strike K and maturity T. Compute the cost of this option. In a hedging portfolio containing only bonds and the asset S, what quantity of the asset should be held at any time ?

Proof. 1. We have $V_t \triangleq \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}V_T|\mathcal{F}_t]$, where $\mathbb{Q}$ is the risk neural probability. Indeed

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = e^{-\frac{1}{2}\lambda^2 t - \lambda W_t},$$

and so $\tilde{W}_t := W_t - \lambda t$ is a $\mathbb{Q}$ brownian motion. And of course $\mathrm{d}S_t = S_t r\,\mathrm{d}t + S_t\sigma\,\mathrm{d}\tilde{W}_t$. Hence,

$$\begin{aligned}
V_t &= \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}yS_t^\gamma e^{\gamma(r-\frac{1}{2}\sigma^2)(T-t)+\gamma\sigma(\tilde{W}_T-\tilde{W}_t)}|\mathcal{F}_t] \\
&= e^{(\gamma-1)r(T-t)}e^{-\frac{1}{2}\sigma^2(T-t)}yS_t^\gamma\mathbb{E}_{\mathbb{Q}}[e^{\gamma\sigma\sqrt{T-t}Z}|\mathcal{F}_t], \qquad \text{where } Z \sim \mathcal{N}(0,1) \\
&= e^{(\gamma-1)r(T-t)}e^{-\frac{1}{2}\sigma^2(T-t)}yS_t^\gamma e^{\frac{1}{2}\gamma^2\sigma^2(T-t)} \\
&= yS_t^\gamma A(t,T),
\end{aligned}$$

with $A(t,T) = e^{(\gamma-1)(T-t)(r+\frac{1}{2}\gamma\sigma^2)}$.

For the portfolio composition we know that $\mathrm{d}V_t = \delta_t\,\mathrm{d}S_t + \delta_t^0\,\mathrm{S}_t^0$ where each part is the amount of risky or non risky asset.

$$\begin{aligned}
\mathrm{d}V_t &= yA(t,T)\,\mathrm{d}(S_t^\gamma) + yS_t^\gamma\,\mathrm{d}(A(t,T)) + \underbrace{\mathrm{d}\langle S^\gamma, A(\cdot,T)\rangle_t}_{=0} \\
&= yA(t,T)\left(\gamma S_t^{\gamma-1}\,\mathrm{d}S_t + \frac{1}{2}\gamma(\gamma-1)S_t^{\gamma-1}S_t^2\sigma^2\,\mathrm{d}t\right) + yS_t^\gamma(1-\gamma)(r+\frac{1}{2}\gamma\sigma^2)A(t,T)\,\mathrm{d}t \\
&= yA(t,T)\gamma S_t^{\gamma-1}\,\mathrm{d}S_t + yS_t^\gamma(\gamma-1)rA(t,T)\,\mathrm{d}t.
\end{aligned}$$

By identifying and with the fact that $\mathrm{d}S_t^0 = rS_t^0\,\mathrm{d}t$, we have

$$\begin{aligned}
\delta_t &= yA(t,T)\gamma S_t^{\gamma-1}; \\
\delta_t^0 &= \frac{yS_t^\gamma(\gamma-1)}{S_t^0}.
\end{aligned}$$

2. We already made a very similar calculus,

$$\begin{aligned} \mathrm{d}X_t &= y\,\mathrm{d}(S_t^\gamma) \\ &= y\left(\gamma S_t^{\gamma-1}\,\mathrm{d}S_t + \frac{1}{2}\gamma(\gamma-1)S_t^{\gamma-1}S_t^2\sigma^2\,\mathrm{d}t\right) \\ &= X_t\gamma\left((r+\frac{1}{2}(\gamma-1)\sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}\tilde{W}_t\right). \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{d}V_t &= X_t\,\mathrm{d}(A(t,T)\ +\ A(t,T)\,\mathrm{d}X_t \\ &= X_t(1-\gamma)(r+\frac{1}{2}\gamma\sigma^2)A(t,T)\,\mathrm{d}t\ +\ A(t,T)X_t\gamma((r+\frac{1}{2}(\gamma-1)\sigma^2)\,\mathrm{d}t+\sigma\,\mathrm{d}\tilde{W}_t) \\ &= V_t(r\,\mathrm{d}t+\sigma\gamma\,\mathrm{d}\tilde{W}_t). \end{aligned}$$

3. We have

$$\begin{aligned} P_t &\triangleq \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(K-V_T)^+|\mathcal{F}_t] \\ &= Ke^{-r(T-t)}\Phi(-d^-)\ -\ V_t\Phi(-d^+), \end{aligned}$$

with

$$d^{\pm} := \frac{\ln\frac{V_t}{Ke^{-r(T-t)}}}{\sigma\gamma\sqrt{T-t}} \pm \frac{1}{2}\sigma\gamma\sqrt{T-t}.$$

We have $\delta_t^P = -\Phi(-d^+)$. So the hedging of the portfolio

$$\begin{aligned} \mathrm{d}P_t &= \delta_t^P\,\mathrm{d}(V_t)\ +\ \cdot\,\mathrm{d}t \\ &= \delta_t^P V_t\frac{\mathrm{d}V_t}{V_t}\ +\ \cdot\,\mathrm{d}t \\ &= \delta_t^P yA(t,T)S_t^\gamma\sigma\gamma\,\mathrm{d}\tilde{W}_t\ +\ \cdot\,\mathrm{d}t, \qquad \text{and } \sigma\mathrm{d}\tilde{W}_t = \frac{\mathrm{d}S_t}{S_t} - r\,\mathrm{d}t \\ &= \delta_t^P yA(t,T)S_t^{\gamma-1}\gamma\,\mathrm{d}S_t\ +\ \cdot\,\mathrm{d}t. \end{aligned}$$

So we have to buy $-\Phi(-d^+)yA(t,T)S_t^{\gamma-1}\gamma$ in the asset S.

□