Stochastic processes and derivatives
Sheet 3

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Exercise 1 (Binary option)**.** We derive the price of a binary option from the call price. The hedging strategy gives exploding Delta at the money and practitioners prefer to use static replicating strategy using a spread of call options. This is illustrated in this exercise.

Assume that the rate of interest is a constant value $r \geq 0$. The value of a stock follows the dynamic

$$\mathrm{d}S_t \quad = \quad S_t r \, \mathrm{d}t \; + \; \sigma \, \mathrm{d}W_t \; , \qquad S_0 = x.$$

A binary option pays out 1 if the stock price is greater than or equal to the strike K at maturity T, 0 otherwise.

1. Compute the price of the binary option at any time t. Use the usual notation of the Black–Scholes pricing formula.
2. Compute the Delta of this option and describe the dynamic strategy for the replications portfolio of the option. In particular, what happens to the number of stocks in the portfolio as the time to maturity goes to zero at the money ?
3. Let $\varepsilon \in]0, K[$. For maturity T, consider a static portfolio in which the holder has longed ε^{-1} calls with strike $K - \varepsilon$ and shorted ε^{-1} calls with strike K at time 0. Prove that this is a superhedging portfolio for the binary option, *i.e.* the value of the portfolio at maturity exceeds the payoff of the binary.

Proof. 1. The payoff is $h(x) = \mathbb{1}_{\{x \geq K\}}$.

$$\begin{aligned} BinC_t(x, K) &\triangleq \mathbb{E}[e^{-r(T-t)} h(x)] \\ &= -\mathbb{E}[e^{-r(T-t)} \partial_K (x - K)^+] \\ &= -\partial_K C(x, K) \\ &= -\partial_K \left(S_0 \Phi(d^+) - K e^{-r(T-t)} \Phi(d^-) \right) \\ &= e^{-r(T-t)} \Phi \left(\frac{\ln \left(\frac{S_0}{K e^{-r(T-t)}} \right)}{\sigma \sqrt{T-t}} - \frac{1}{2} \sigma \sqrt{T-t} \right). \end{aligned}$$

2. We deduce easily the Delta

$$\begin{aligned} \Delta \left(BinC_t(x, K) \right) &\triangleq \frac{\partial e^{-r(T-t)} \Phi(d^-)}{\partial S_0} \\ &= \frac{e^{-r(T-t)}}{S_0 \sigma \sqrt{T-t}} \phi \left(\frac{\ln \left(\frac{S_0}{K e^{-r(T-t)}} \right)}{\sigma \sqrt{T-t}} - \frac{1}{2} \sigma \sqrt{T-t} \right). \end{aligned}$$

The strategy for replicating this option is to buy Delta in the underlying and short the rest of the portfolio value minus delta times the sport of the underlying in cash (to satisfy the autofinancing property).

In particular at the money the Delta is equal to $\frac{e^{-r(T-t)}}{S_0 \sigma \sqrt{T-t}} \phi \left(d^- \right) \xrightarrow[t \to T]{} \infty$. So one had to buy an infinity of the underlying to hedge his position.

3. We have the value of the portfolio at T:

$$\begin{aligned}\Pi_T &= \varepsilon^{-1}C_T(S_T, K-\varepsilon) \;-\; \varepsilon^{-1}C_T(S_T, K)\\ &= \begin{cases} 0 & \text{if } S_T < K-\varepsilon\ ,\\ \varepsilon^{-1}(S_T - K + \varepsilon) & \text{if } K-\varepsilon \le S_T < K\ ,\\ 1 & \text{else.}\end{cases}\\ &\ge BinC_T(S_T, K).\end{aligned}$$

□

Exercise 2 (Pay-later option)**.** A pay-later option is one which pays out like a regular call option but charges a predetermined fee P at the maturity rather than at the signing date, and this fee is only paid if the stock price exceeds the strike at maturity. This implies that the owner of the option pays nothing at the signing date, and must only pay the fee P if the option is in the money at maturity.

Assume that the rate of interest is a constant value $r \ge 0$. The value of a stock follows the dynamic

$$\mathrm{d}S_t \;=\; S_t r\,\mathrm{d}t \;+\; \sigma\,\mathrm{d}W_t\ , \qquad S_0 = x.$$

1. Write down the no-arbitrage condition and use it to compute the fee that should be charged for a pay-later option. Prove that the fee of the pay-later is greater than the price of a usual call.
2. Suggest a static portfolio containing classical financial instruments that would replicate the pay-later option.
3. What is the delta of this option.

Proof. 1. We have the payoff $h(x) = (x-K)^+ - P\mathbb{1}_{\{x>K\}}$. So we have with the AOA

$$\begin{aligned} & \mathbb{E}[e^{-rT}(x-K)^+ - e^{-rT}P\mathbb{1}_{\{x>K\}}] \;=\; 0\\ \Leftrightarrow\quad & C_0(x,K,T) - P.BinC_0(x,K,T) \;=\; 0\\ \Leftrightarrow\quad & P \;=\; \frac{C_0(x,K,T)}{BinC_0(x,K,T)} \;>\; C_0(x,K,T).\end{aligned}$$

2. An easy strategy is to be long one call of strike K and maturity T and to short P binary call of strike K and maturity T.

3. The delta of such an option is the sum of the delta of a call option and P times the delta of a binary call:

$$\Delta_t \;=\; \Delta(C_t) + P\Delta(BinC_t).$$

□

Exercise 3 (Forward-start call option)**.** Assume that the interest rate is a constant value $r \ge 0$. A forward start call with maturity T and parameter $\theta \in]0, T[$ is an option that pays out $(S_T - S_\theta)^+$ at the time T, where S is the price of the underlying following the BS model.

1. What is the value of the option in the time interval $[\theta, T]$?
2. What is the value of the option in the time interval $[0, \theta]$? Describe the hedging strategy on $[0, T]$.
3. Compute the Delta and the Gamma of the option.

Proof. 1. At $t \in [\theta, T]$, S_θ is known so the price of the option is the price of a call of maturity T and strike S_θ:

$$FSC_t \;=\; S_t\Phi(d^+) - e^{-r(T-t)}S_\theta\Phi(d^-)\,,$$

where

$$d^\pm \;=\; \frac{\ln\left(\frac{S_t}{S_\theta e^{-r(T-t)}}\right)}{\sigma\sqrt{T-t}} \pm \frac{1}{2}\sigma\sqrt{T-t}.$$

2. When $t \in [0, \theta]$,

$$\begin{aligned}
FSC_t &\triangleq \mathbb{E}[e^{-r(T-t)}(S_T - S_\theta)^+|\mathcal{F}_t] \\
&= \mathbb{E}\left[e^{-r(\theta-t)}\mathbb{E}[e^{-r(T-\theta)}(S_t - S_\theta)^+|\mathcal{F}_\theta]|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[e^{-r(\theta-t)}\left(S_\theta\Phi\left(\frac{(r+\frac{\sigma^2}{2})\sqrt{T-\theta}}{\sigma}\right) - S_\theta e^{-r(T-\theta)}\Phi\left(\frac{(r-\frac{\sigma^2}{2})\sqrt{T-\theta}}{\sigma}\right)\right)|\mathcal{F}_t\right] \\
&= \mathbb{E}[e^{-r(\theta-t)}S_\theta\Lambda|\mathcal{F}_t]
\end{aligned}$$

And we recall that $S_\theta = S_t e^{r(\theta-t)-\frac{\sigma^2}{2}(\theta-t)+\sigma(W_\theta - W_t)}$. So,

$$\begin{aligned}
FSC_t &= \Lambda S_t e^{-\frac{\sigma^2}{2}(\theta-t)}\mathbb{E}[e^{\sigma(W_\theta-W_t)}|\mathcal{F}_t] \\
&= \Lambda S_t.
\end{aligned}$$

So we have the prix at any time $t \in [0, T]$, $FSC_t = \Lambda S_T \mathbb{1}_{[0,\theta]}(t) + C(t, S_t, S_\theta, T)\mathbb{1}_{[\theta,T]}(t)$. With this decomposition of the price, the hedging strategy is immediate.

3. We clearly see

$$\begin{aligned}
\Delta(FSC_t) &= \Lambda + \Delta(C_t)\ ; \\
\Gamma(FSC_t) &= \Gamma(C_t).
\end{aligned}$$

□