# Stochastic processes and derivatives
# Sheet 2

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**Exercise 1** (Stochastic volatility)**.** Assume that the market offers interest rate 0. At time $T > 0$, consider a stock whose value is $S_T = S_0 \exp\left(-\frac{1}{2}\tilde{\sigma}^2 T + \tilde{\sigma} W_T\right)$, where $S_0 > 0$, $(W_t)_{t \geq 0}$ is a Brownian motion, and $\tilde{\sigma}$ is a r.v. independent of $W_T$.

1. Let $K > 0$ be a fixed strike. Determine explicitly the deterministic function $h : \mathbb{R} \to \mathbb{R}$ s.t. $C_0 := \mathbb{E}[(S_T - K)^+] = \mathbb{E}[h(\tilde{\sigma}^2)]$.
2. Discuss the accuracy of the approximation $C_0 \approx h(\mathbb{E}[\tilde{\sigma}^2]) + \frac{1}{2}\mathrm{Var}[\tilde{\sigma}^2]\partial_x^2 h(x)|_{x=\mathbb{E}[\tilde{\sigma}^2]}$.
3. Show that $C_0 < C^{BS} := h(\mathbb{E}[\tilde{\sigma}^2])$ around the money, and $C_0 > C^{BS}$ in/out the money. Paying particular to $\sigma_I = \sqrt{\mathbb{E}[\tilde{\sigma}^2]}$.

*Proof.* 1. We have,

$$\begin{aligned} C_0 &\triangleq \mathbb{E}\left[\left(S_0 e^{-\frac{1}{2}\tilde{\sigma}^2 T + \tilde{\sigma} W_T} - K\right)^+\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\left(S_0 e^{-\frac{1}{2}\tilde{\sigma}^2 T + \tilde{\sigma} W_T} - K\right)^+ \middle| \tilde{\sigma}^2\right]\right] \\ &= \mathbb{E}\left[Call^{BS}(T, K, S_0, \tilde{\sigma}^2)\right]. \end{aligned}$$

So we identify $h$:

$$\begin{aligned} h(x) &= S_0 \Phi(d_+(x)) - K\Phi(d_-(x)), \qquad \text{with} \\ d_\pm(x) &= \frac{\ln\left(\frac{S_0}{K}\right)}{\sqrt{Tx}} \pm \frac{1}{2}\sqrt{Tx}. \end{aligned}$$

2. Let us do a Taylor expansion of second order of $h(\tilde{\sigma}^2)$ around $\mathbb{E}[\tilde{\sigma}^2]$,

$$h\left(\tilde{\sigma}^2\right) = h\left(\mathbb{E}[\tilde{\sigma}^2]\right) + \left(\tilde{\sigma}^2 - \mathbb{E}[\tilde{\sigma}^2]\right)\partial_x h(x)|_{x=\mathbb{E}[\tilde{\sigma}^2]} + \frac{1}{2}\left(\tilde{\sigma}^2 - \mathbb{E}[\tilde{\sigma}^2]\right)^2 \partial_x^2 h(x)|_{x=\mathbb{E}[\tilde{\sigma}^2]} + \varepsilon.$$

Now taking the expectation we have

$$\begin{aligned} C_0 &\triangleq \mathbb{E}\left[h\left(\tilde{\sigma}^2\right)\right] \\ &= h(\mathbb{E}[\tilde{\sigma}^2]) + \frac{1}{2}\mathrm{Var}[\tilde{\sigma}^2]\partial_x^2 h(x)|_{x=\mathbb{E}[\tilde{\sigma}^2]} + \varepsilon, \end{aligned}$$

where $\varepsilon$ is an error. This approximation works in the case were $\tilde{\sigma}^2$ is stochastic but "not so much", and so $\tilde{\sigma}^2$ is near form its expected value.

3. If we consider that the previous approximation is an equality we have $C_0 = C^{BS} = \frac{1}{2}\mathrm{Var}[\tilde{\sigma}^2]\partial_x^2 h(x)|_{x=\mathbb{E}[\tilde{\sigma}^2]}$. Let us check the sign of the second order derivative of $h$.

$$\begin{aligned} \partial_x h(x) &= S_0 \phi(d_+(x)) d'_+(x) - K\phi(d_-(x)) d'_-(x) \\ &= \frac{K}{\sqrt{2\pi}} e^{-\frac{d_-^2(x)}{2}} \frac{1}{2}\sqrt{\frac{T}{x}}, \end{aligned}$$

we have used the fact that $S_0\phi(d_+) = K\phi(d_-)$. And then after calculations

$$\partial_x^2 h(x) \quad = \quad \frac{K}{2\sqrt{2\pi}} e^{-\frac{d_-^2(x)}{2}} \sqrt{\frac{T}{x}} \left( \frac{1}{2x^2 T} \left( \ln \frac{S_0}{K} \right)^2 - \frac{T}{8} - \frac{1}{2x} \right).$$

So at the money and around, $h''(x) < 0$ and $C_0 < C^{BS}$. And if $K \ll S$ of $K \gg S$ we have $C_0 > C^{BS}$. This shows that we have a smile in the implied volatility.

□

**Exercise 2** (Implicit volatility expansion). 1. Let $\lambda \in \mathbb{C}$. Let $h(x) = (e^k - e^k)^+$. Show that $\hat{h}(\lambda)$, the generalized Fourier transform of $h$ at $\lambda$ with $Im(\lambda) < -1$, *i.e.*

$$\hat{h}(\lambda) \quad := \quad \int_{\mathbb{R}} h(x) e^{i\lambda x} \, \mathrm{d}x$$

is given by $\frac{-e^{k - ik\lambda}}{i\lambda + \lambda^2}$.

2. Show that

$$\begin{aligned} u(t, x, k, \sigma_0) \quad &:= \quad \frac{1}{2\pi} \int_{\mathbb{R}} \hat{h}(\lambda) e^{t\phi_0(\lambda, \sigma_0)} \, \mathrm{d}Re(\lambda) \\ &= \quad e^x \Phi(d_+) \; - \; e^k \Phi(d_-) \, , \end{aligned}$$

where $\phi_0(\lambda, \sigma_0) := -\frac{1}{2}\sigma_0^2(\lambda^2 + i\lambda)$, $d_\pm := \frac{x-k}{\sigma_0\sqrt{t}} \pm \frac{1}{2}\sigma_0\sqrt{t}$, and $\Phi$ is the distribution function of the standard Gaussian.

3. Deduce that $\sigma \mapsto u(t, x, k, \sigma)$ is analytic. Write down the Taylor expansion of $u(t, x, k, \sigma_0 + \delta)$ around $\sigma_0$.

4. Let $\varepsilon > 0$ and define $\phi^\varepsilon(t, \lambda, \sigma_0) := (1 - \varepsilon)t\phi_0(\lambda, \sigma_0) + \varepsilon\phi(t, \lambda)$, where $i\lambda x + \phi(t, \lambda) := \ln\left(\mathbb{E}_x[e^{i\lambda X_t}]\right)$ for some process $X_t$. Next define

$$u^\varepsilon(t, x, k, \sigma_0) \quad := \quad \frac{1}{2\pi} \int_{\mathbb{R}} \hat{h}(\lambda) e^{i\lambda x} e^{\phi^\varepsilon(t, \lambda, \sigma_0)} \, \mathrm{d}Re(\lambda).$$

Thus, $u^\varepsilon(t, x, k, \sigma_0)|_{\varepsilon=1} = u^X$, where $u^X$ is the price of payoff $h$ under the process $X_t$ and $u^\varepsilon(t, x, k, \sigma_0)|_{\varepsilon=1} = u^{BS}$, the price under Black-Scholes formula. Deduce that

$$u^\varepsilon(t, x, k, \sigma_0) \quad = \quad \sum_{k=0}^{\infty} \varepsilon^n u_n(t, x, k, \sigma_0) \, ,$$

where

$$u_n(t, x, k, \sigma_0) \quad := \quad \frac{1}{n! 2\pi} \int_{\mathbb{R}} \hat{h}(\lambda) e^{i\lambda x} e^{t\phi_0(\lambda, \sigma_0)} (\phi(t, \lambda) - t\phi_0(\lambda, \sigma_0))^n \, \mathrm{d}Re(\lambda).$$

*Proof.* 1. We have

$$\begin{aligned} \hat{h}(\lambda) \quad &\triangleq \quad \int_{\mathbb{R}} (e^x - e^k)^+ e^{-i\lambda x} \, \mathrm{d}x \\ &= \quad \int_k^\infty (e^x - e^k) e^{-i\lambda x} \, \mathrm{d}x \\ &= \quad \left[ \frac{1}{1 - i\lambda} e^{x(1-i\lambda)} + \frac{1}{i\lambda} e^{k - i\lambda x} \right]_k^\infty \\ &= \quad \frac{-e^{k - ik\lambda}}{i\lambda + \lambda^2}. \end{aligned}$$

2. We set $X_t := x - \frac{1}{2}\sigma_0^2 t + \sigma_0 W_t$, where $W$ is a Brownian motion. Hence

$$\begin{aligned} e^x\Phi(d_+) - e^k\Phi(d_-) \quad &\triangleq \quad \mathbb{E}_x[h(X_t)] \\ &= \quad \mathbb{E}_x \left[ \frac{1}{2\pi} \int_{\mathbb{R}} \hat{h}(\lambda) e^{i\lambda X_t} \, \mathrm{d}Re(\lambda) \right] \\ &= \quad \frac{1}{2\pi} \int_{\mathbb{R}} \hat{h}(\lambda) e^{i\lambda x} \mathbb{E}_x[e^{-\frac{1}{2}i\lambda\sigma_0^2 t + i\lambda\sigma_0 W_t}] \, \mathrm{d}Re(\lambda). \end{aligned}$$

As $W_t \sim \mathcal{N}(0, t)$ we have $\mathbb{E}_x[e^{-\frac{1}{2}i\lambda\sigma_0^2 t + i\lambda\sigma_0 W_t}] = e^{-\frac{1}{2}i\lambda\sigma_0^2 t + \frac{1}{2}\sigma_0^2\lambda^2 t} = e^{t\phi_0(\lambda, \sigma_0)}$.

3. $\hat{h}$ is analytic and $\phi_0$ as well, so by Fubini $\sigma \mapsto u(t,x,k,\sigma)$ too.

$$u(t,x,k,\sigma_0+\delta) \;=\; \sum_{n\geq 1} \frac{\delta^n}{n!} \left.\frac{\partial^n u}{\partial \sigma^n}\right|_{\sigma=\sigma_0} ,$$

and $\partial_\sigma^n u|_{\sigma=\sigma_0} = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{h}(\lambda)e^{i\lambda x}\partial_\sigma^n e^{t\phi_0(\lambda,\sigma)}|_{\sigma=\sigma_0}\, \mathrm{d}Re(\lambda)$.

□

**Exercise 3** (Bachelier model)**.** Assume the market offers interest rate 0. The Bachelier model for the evolution of a stock price is given by

$$S_t^B \;=\; S_0 \;+\; \sigma^B W_t \;,$$

for $t \geq 0$, where $S_0 \geq 0$ is the spot price, $\sigma^B > 0$, and $(W_t)_{t\geq 0}$ a Brownian motion.

1. Prove that under the Bachelier model, the price of a call option on the stock with strike $K$ at expiry time $T$ is given by

$$C_0^B \;:=\; (S_0-K)\Phi\left(\frac{S_0-K}{\sigma^B\sqrt{T}}\right) \;+\; \sigma^B\sqrt{T}\phi\left(\frac{S_0-K}{\sigma^B\sqrt{T}}\right).$$

2. Let $\sigma^B = S_0\sigma^{BS}$. using the relation $e^y \geq 1+y$, prove that at the money option prices given by the Bachelier and Black-Scholes models satisfy

$$0 \;\leq\; C_0^B - C_0^{BS} \;\leq\; \frac{S_0}{24\sqrt{2\pi}}\left(\sigma^{BS}\right)^3 T^{\frac{3}{2}}.$$

   Comment ont the accuracy of the Bachelier formula as an estimator of the Black-Scholes formula for the option price.

3. Suppose that one knows the option price is $C_0$ (model independent). Show that the at the money implied volatility $\sigma_I^B$ and $\sigma_i^{BS}$ yielded respectively by the Balchelier and Balck-Scholes models satisfy

$$0 \;\leq\; \sigma_I^{BS} - \frac{\sigma_I^B}{S_0} \;\leq\; \frac{\left(\sigma_i^{BS}\right)^3 T}{24}.$$

   Comment on the accuracy of the Bachelier implied volatility as an estimator for the Black-Scholes implied volatility.

*Proof.* 1.

$$\begin{aligned}
C_0^B &\triangleq \mathbb{E}[(S_0+\sigma^B W_T - K)^+] \\
&= \int_{-d}^{\infty} (S_0+\sigma^B\sqrt{T}z-K)\phi(z)\,\mathrm{d}z\;, \qquad \text{where } d := \frac{S_0-K}{\sigma^B\sqrt{T}} \\
&= (S_0-K)\int_{-d}\infty\phi(z)\,\mathrm{d}z \;+\; \sigma^B\sqrt{T}\int_{-d}\infty z\phi(z)\,\mathrm{d}z \\
&= (S_0-K)(1-\Phi(-d)) \;+\; \sigma^B\sqrt{T}[-\phi(z)]_{-d}^{\infty} \\
&= (S_0-K)\Phi(d) \;+\; \sigma^B\sqrt{T}\phi(d).
\end{aligned}$$

2. At the money we have $C_0^B = \frac{\sigma^B\sqrt{T}}{\sqrt{2\pi}}$. And

$$\begin{aligned}
C_0^{BS} &= \frac{S_0}{\sqrt{2\pi}}\int_{-\frac{\sigma^{BS}\sqrt{T}}{2}}^{\frac{\sigma^{BS}\sqrt{T}}{2}} e^{-\frac{x^2}{2}}\,\mathrm{d}x \\
&\leq \frac{S_0}{\sqrt{2\pi}}\sigma^{BS}\sqrt{T} \;=\; C_0^B.
\end{aligned}$$

On the other hand we have

$$
\begin{aligned}
C_0^{BS} &\geq \frac{S_0}{\sqrt{2\pi}} \int_{-\frac{\sigma^{BS}\sqrt{T}}{2}}^{\frac{\sigma^{BS}\sqrt{T}}{2}} 1 + \frac{x^2}{2} \,\mathrm{d}x \\
&\geq C_0^B - \frac{S_0}{\sqrt{2\pi}} \left[\frac{x^3}{6}\right]_{-\frac{\sigma^{BS}\sqrt{T}}{2}}^{\frac{\sigma^{BS}\sqrt{T}}{2}} \\
&\geq C_0^B - \frac{S_0}{24\sqrt{2\pi}} \left(\sigma^{BS}\sqrt{T}\right)^3 .
\end{aligned}
$$

Which gives us the control on the difference between the pricing of the two models. When the volatility is low, Bachelier is a good approximation of BS.

3. we have the following relations

$$
\begin{aligned}
C_0 &= \frac{\sigma_I^B \sqrt{T}}{\sqrt{2\pi}} \\
&= \frac{S_0}{\sqrt{2\pi}} \int_{-\frac{\sigma_I^{BS}\sqrt{T}}{2}}^{\frac{\sigma_I^{BS}\sqrt{T}}{2}} e^{-\frac{x^2}{2}} \,\mathrm{d}x.
\end{aligned}
$$

Then we use the same inequalities as in the previous question to deduce the relation.

□