Stochastic processes and derivatives
Sheet 1

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Exercise 1 (Process due to Hamza and Klebaner)**.** Let $(B_t, \tilde{B}_t, W_t)_{t\geq 0}$ be three independent Brownian motions, and define the process

$$X_t = \begin{cases} B_t & \text{for } t \geq 1 \;, \\ \sqrt{t}\left(B_1 \cos W_{\ln t} + \tilde{B}_1 \sin W_{\ln t}\right) & \text{for } t < 1. \end{cases}$$

1. Show that $\mathbb{E}[X_t] = 0$ and that $\mathrm{Var}[X_t] = t$ for all $t \geq 0$.

2. Show that for any fixes $t \geq 0$, X_t is normally distributed.

3. Define the filtration

$$\mathcal{F}_t = \begin{cases} \sigma(B_u : u \leq t) & \text{for } t \geq 1 \;, \\ \sigma(B_u, \tilde{B}_u, W_v : u \leq 1, v \leq \ln t) & \text{for } t < 1. \end{cases}$$

   Show that for any $0 \leq s \leq t$, $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$.

Proof. 1. We have for $t \geq 0$,

$$\begin{aligned} \mathbb{E}[X_t] &= \mathbb{E}\left[B_t \mathbb{1}_{t\in[0,1]} \;+\; \sqrt{t}(B_1 \cos W_{\ln t} + \tilde{B}_1 \sin W_{\ln t})\mathbb{1}_{t\in[1,\infty[}\right] \\ &= \mathbb{E}[B_t]\mathbb{1}_{t\in[0,1]} \;+\; \sqrt{t}\left(\mathbb{E}[B_1]\mathbb{E}[\cos W_{\ln t}] + \mathbb{E}[\tilde{B}_1]\mathbb{E}[\sin W_{\ln t}]\right)\mathbb{1}_{t\in[1,\infty[} \qquad \text{as they are independent} \\ &= 0 \qquad \text{as } \mathbb{E}[B_s] = 0 \text{ for all } s \geq 0. \end{aligned}$$

For $t \leq 1$, $\mathrm{Var}[X_t] = \mathrm{Var}[B_t] = t$. For $t > 1$,

$$\begin{aligned} \mathrm{Var}[X_t] &= \mathrm{Var}\left[\sqrt{t}(B_1 \cos W_{\ln t} + \tilde{B}_1 \sin W_{\ln t})\right] \\ &= t\left(\mathrm{Var}[B_1 \cos W_{\ln t}] + \mathrm{Var}[\tilde{B}_1 \sin W_{\ln t}] + 2\mathrm{Cov}(B_1 \cos W_{\ln t}, \tilde{B}_1 \sin W_{\ln t})\right) \\ &= t\left(\mathbb{E}[B_1^2 \cos^2 W_{\ln t}] + \mathbb{E}[\tilde{B}_1^2 \sin^2 W_{\ln t}]\right) \\ &= t\left(\mathbb{E}[B_1^2]\mathbb{E}[\cos^2 W_{\ln t}] + \mathbb{E}[\tilde{B}_1^2]\mathbb{E}[\sin^2 W_{\ln t}]\right) \\ &= t\mathbb{E}[\cos^2 W_{\ln t} + \sin^2 W_{\ln t}] \;=\; t. \end{aligned}$$

2. For $t \leq 1$, $X_t = B_t \sim \mathcal{N}(0,t)$. For $t > 1$ let us compute the Fourier transform of X_t,

$$\begin{aligned} \mathbb{E}\left[e^{iuX_t}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{iu\sqrt{t}(B_1 \cos W_{\ln t} + \tilde{B}_1 \sin W_{\ln t})}|W_{\ln t}\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{iu\sqrt{t}B_1 \cos W_{\ln t}}|W_{\ln t}\right]\mathbb{E}\left[e^{iu\tilde{B}_1 \sin W_{\ln t}}|W_{\ln t}\right]\right] \\ &= \mathbb{E}\left[e^{-\frac{u^2 t \cos^2 W_{\ln t}}{2}} e^{-\frac{u^2 t \sin^2 W_{\ln t}}{2}}\right] \\ &= e^{-\frac{u^2 t}{2}} \;\sim\; \mathbb{E}\left[e^{iu\mathcal{N}(0,t)}\right]. \end{aligned}$$

3. For $0 \le s \le t \le 1$, $\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[B_t|\mathcal{F}_s] = \mathbb{E}[B_t - B_s + B_s|\mathcal{F}_s] = \mathbb{E}[B_{t-s}|\mathcal{F}_s] + B_s = B_s = X_s$.
For $1 < s \le t$,

$$\begin{aligned}
\mathbb{E}[X_t|\mathcal{F}_s] &= \mathbb{E}\left[\sqrt{t}\left(B_1 \cos(W_{\ln t} - \cos W_{\ln s} + \cos W_{\ln s}) + \tilde{B}_1(\sin W_{\ln t} - \sin W_{\ln s} + \sin W_{\ln s})\right)\Big| \mathcal{F}_s\right] \\
&= \mathbb{E}\Big[\sqrt{t}B_1 \left(\cos(W_{\ln t} - W_{\ln s})\cos W_{\ln s} - \sin(W_{\ln t} - W_{\ln s})\sin W_{\ln s}\right) \\
&\qquad + \sqrt{t}\tilde{B}_1 \left(\sin(W_{\ln t} - W_{\ln s})\cos W_{\ln s} + \cos(W_{\ln t} - W_{\ln s})\sin W_{\ln s}\right)\Big| \mathcal{F}_s\Big] \\
&= \sqrt{t}B_1 \left(\cos W_{\ln s}\mathbb{E}\left[\cos(W_{\ln t - \ln s})\right] - \sin W_{\ln s}\mathbb{E}\left[\sin(W_{\ln t - \ln s})\right]\right) \\
&\qquad + \sqrt{t}\tilde{B}_1 \left(\cos W_{\ln s}\mathbb{E}\left[\sin(W_{\ln t - \ln s})\right] - \sin W_{\ln s}\mathbb{E}\left[\cos(W_{\ln t - \ln s})\right]\right) \\
&= \sqrt{t}B_1 \cos W_{\ln s} e^{-\frac{\ln t - \ln s}{2}} - 0 + 0 + \sqrt{t}\tilde{B}_1 \sin W_{\ln s} e^{-\frac{\ln t - \ln s}{2}} \\
&= \sqrt{s}\left(B_1 \cos W_{\ln s} + \tilde{B}_1 \sin W_{\ln s}\right) \;=\; X_s.
\end{aligned}$$

We use the same technique for $0 \le s \le 1 < t$. $\square$

Exercise 2 (Formula of Brenner and Subrahmanyan)**.** The Black-Scholes pricing formula of a call option with strike K at maturity T on a stock with volatility σ is given by

$$C^{BS}(T, K, S) \;=\; S\Phi(d_1) - Ke^{-rT}\Phi(d_2)\ ,$$

where S is the current price of the underlying, r the interest rate and Φ the standard Gaussian distribution. And,

$$\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S}{Ke^{-rT}}\right) + \sigma^2\frac{T}{2}}{\sigma\sqrt{T}}\ ; \\
d_2 &= d_1 - \sigma\sqrt{T}.
\end{aligned}$$

Using a first Taylor expansion of $\Phi(x)$ around $x = 0$, deduce that

$$C^{BS}(T, K, S) \;\approx\; 0.4S\sigma\sqrt{T}\ ,$$

for $S = Ke^{-rT}$. With the same technique show that the Delta of the call is approximately $0.5 + 0.2\sigma\sqrt{T}$ for $S = Ke^{-rT}$. Finally, show that the Vega of the call is approximately $0.4\sqrt{T}Se^{-\sigma^2\frac{T}{8}}$ for $S = Ke^{-rT}$.

Proof. With $S = Ke^{-rT}$ we have $d_1 = \frac{1}{2}\sigma\sqrt{T}$, $d_2 = -\frac{1}{2}\sigma\sqrt{T}$ and $C^{BS}(T, K, S) = S(\Phi(d_1) - \Phi(d_2))$. We write the first order Taylor expansion around $x = 0$,

$$\Phi(d_1) \;\approx\; \Phi(0) \;+\; d_1\phi(0)\ , \qquad \text{when } \sigma\sqrt{T} \to 0.$$

Where $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. So we have

$$\begin{aligned}
C^{BS}(T, K, S) &\approx S(\Phi(0) \;+\; d_1\phi(0) \;-\; \Phi(0) \;-d_2\phi(0)) \\
&\approx S\sigma\sqrt{T}\underbrace{\frac{1}{\sqrt{2\pi}}}_{\approx 0.4}.
\end{aligned}$$

We know that $\Delta^{BS}(C) = \Phi(d_1) \approx \Phi(0) + d_1\phi(0)$ when $S = Ke^{-rT}$ and $\sigma\sqrt{T} \to 0$. So $\Delta^{BS}(C) \approx \frac{1}{2} + 0.2\sigma\sqrt{T}$. When $S = Ke^{-rT}$, $C^{BS}(T, K, S) = S(\Phi(\frac{1}{2}\sigma\sqrt{T}) - \Phi(-\frac{1}{2}\sigma\sqrt{T})$. So the Vega is

$$\begin{aligned}
\nu^{BS}(C) &\triangleq \frac{\partial C^{BS}(T, K, S)}{\partial\sigma} \\
&= S\left(\frac{1}{2}\sqrt{T}\ \phi\left(\frac{1}{2}\sigma\sqrt{T}\right) \;+\; \frac{1}{2}\sqrt{T}\ \phi\left(-\frac{1}{2}\sigma\sqrt{T}\right)\right) \\
&= S\sqrt{T}\frac{1}{2\pi}e^{-\frac{1}{2}\frac{1}{4}\sigma^2 T} \\
&\approx 0.4\sqrt{T}Se^{-\sigma^2\frac{T}{8}}.
\end{aligned}$$

$\square$

Exercise 3 (Pricing with dividends)**.** For a right continuous process $(S_t)_{t\geq 0}$, we define $S_t- := \lim_{r\uparrow t} S_r$. Suppose that a stock with price process $(S_t)_{t\geq 0}$ pay a dividend $\delta_1 + y_1 S_{t_1-}$ for $\delta_1 > 0$ and $y_1 \in [0,1[$ at fixed time $0 < t_1 < T$. Before and after the dividend payment, the price of the stock evolves like a geometric Brownian motion with zero drift and volatility σ, *i.e.*

$$S_t \;=\; \begin{cases} S_0 \exp\left(-\frac{\sigma^2}{2}t + \sigma W_t\right) & \text{for } t < t_1 \;, \\ S_{t_1} \exp\left(-\frac{\sigma^2}{2}(t-t_1) + \sigma(W_t - W_{t_1})\right) & \text{for } t \geq t_1 . \end{cases}$$

The rate of interest is 0.

1. Write down the stock price S_{t_1} at a payment date t_1 with respect to the price just before the payement S_{t_1-} and the dividend payment and show that

$$\begin{aligned} S_T &= (1-y_1)S_T^{(0)} - \delta_1 \frac{S_T^{(0)}}{S_{t_1}^{(0)}} \\ &= \bar{S}_T^{(0)} - \delta_1 \left(\frac{S_T^{(0)}}{S_{t_1}^{(0)}} - 1\right) , \end{aligned}$$

   where $S_T^{(0)} = S_0 \exp\left(-\frac{\sigma^2}{2}T + \sigma W_T\right)$ is the price of the fictional stock with zero dividends and $\bar{S}_T^{(0)} = (1 - y_1)S_T^{(0)} - \delta_1$.

2. How would you compute the price of a call option with strike K at expiry T on the fictional stock $\bar{S}_T^{(0)}$?

3. Consider an option on the stock with payoff $h(x-K)$ at expiry T, where $h : \mathbb{R} \to [0,\infty[$ is twice differentiable with bounded derivatives.
   Show that $\mathbb{E}[h'(\bar{S}_T^{(0)} - K)] = -\partial_k \mathbb{E}[h(\bar{S}_T^{(0)} - k)]|_{k=K}$.

4. Using a conditioning argument, explicit computations with the Gaussian density, and a change of variable argument, show that,

$$\mathbb{E}\left[h'(\bar{S}_T^{(0)} - K)\frac{S_T^{(0)}}{S_{t_1}^{(0)}}\right] \;=\; -\partial_k \mathbb{E}\left[h\left((1-y_1)e^{\sigma^2(T-t_1)}S_T^{(0)} - k\right)\right]\Big|_{k=K=\delta_1}$$

5. Write down the first order Taylor expansion of $h(S_T - K) - h(\bar{S}_T^{(0)} - K)$. Take expectations in this expansion to derive an approximation of the price of the option with payoff $h(S_T - K)$ in terms of $\mathbb{E}[h(\bar{S}_T^{(0)} - K)]$, the terms computed in parts 3 and 4 and an error term.

Proof. 1. With the condition of AOA we have $S_{t_1} = S_{t_1-} - \delta_1 - y_1 S_{t_1-}$

$$\begin{aligned} S_T &\triangleq S_{t_1} \exp\left(-\frac{\sigma^2}{2}(T-t_1) + \sigma(W_T - W_{t_1})\right) \\ &= S_0 \exp\left(-\frac{\sigma^2}{2}t_1 + \sigma W_{t_1}\right)(1-y_1)\exp\left(-\frac{\sigma^2}{2}(T-t_1) + \sigma(W_T - W_{t_1})\right) \;-\; \delta_1 \exp\left(-\frac{\sigma^2}{2}(T-t_1) + \sigma(W_T - W_{t_1})\right) \\ &= (1-y_1)S_0\exp\left(-\frac{\sigma^2}{2}T + \sigma W_T\right) \;-\; \delta_1 \frac{S_0 \exp\left(-\frac{\sigma^2}{2}T + \sigma W_T\right)}{S_0\exp\left(-\frac{\sigma^2}{2}t_1 + \sigma W_{t_1}\right)} \\ &= (1-y_1)S_T^{(0)} - \delta_1 \frac{S_T^{(0)}}{S_{t_1}^{(0)}}. \end{aligned}$$

2. We are looking for the value of $\mathbb{E}[(\bar{S}_T^{(0)} - K)^+]$,

$$\begin{aligned} \mathbb{E}[(\bar{S}_T^{(0)} - K)^+] &= \mathbb{E}\left[\left((1-y_1)S_0\exp\left(-\frac{\sigma^2}{2}T + \sigma W_T\right) - \delta_1 - K\right)^+\right] \\ &= (1-y_1)\mathbb{E}\left[\left(S_T^{(0)} - \frac{K-\delta_1}{1-y_1}\right)^+\right] \\ &= (1-y_1)\; C^{BS}\left(T, \frac{K-\delta_1}{1-y_1}, S_0\right). \end{aligned}$$

3.

$$
\begin{aligned}
\mathbb{E}[h'(\bar{S}_T^{(0)} - K)] &= \mathbb{E}[h'(g(W_T) - K)] \\
&= \int h'(g(x) - K)\varphi_{W_T}(x)\, \mathrm{d}x \\
&= -\partial_k \left(\int h'(g(x) - K)\varphi_{W_T}(x)\, \mathrm{d}x \right)\Big|_{k=K} \qquad \text{with the dominated convergence} \\
&= -\partial_k \mathbb{E}[h(\bar{S}_T^{(0)} - K)]|_{k=K}.
\end{aligned}
$$

4.

$$
\begin{aligned}
\mathbb{E}\left[h'(\bar{S}_T^{(0)} - K)\frac{S_T^{(0)}}{S_{t_1}^{(0)}}\right] &= \mathbb{E}\left[h'((1-y_1)\frac{S_T^{(0)}}{S_{t_1}^{(0)}}S_{t_1}^{(0)} - K^\delta)\frac{S_T^{(0)}}{S_{t_1}^{(0)}}\right], \qquad \text{where } K^\delta := K - \delta_1 \\
&= \mathbb{E}\left[\mathbb{E}_{W_{t_1}}\left[h'\left((1-y_1)e^{\sigma(W_T - W_{t_1}) - \frac{\sigma^2}{2}(T-t_1)}S_{t_1}^{(0)} - K^\delta\right)e^{\sigma(W_T - W_{t_1}) - \frac{\sigma^2}{2}(T-t_1)}\right]\right] \\
&= \mathbb{E}\left[\int h'\left((1-y_1)e^{\sigma\sqrt{T-t_1}x - \frac{\sigma^2}{2}(T-t_1)}S_{t_1}^{(0)} - K^\delta\right)e^{\sigma\sqrt{T-t_1}x - \frac{\sigma^2}{2}(T-t_1)}\phi(x)\, \mathrm{d}x\right] \\
&= \mathbb{E}\left[\int h'\left((1-y_1)e^{\sigma_1 x - \frac{\sigma_1^2}{2}}S_{t_1}^{(0)} - K^\delta\right)e^{\sigma_1 x - \frac{\sigma_1^2}{2}}\phi(x)\, \mathrm{d}x\right], \qquad \text{where } \sigma_1 := \sigma\sqrt{T-t_1} \\
&= \mathbb{E}\left[\int h'\left((1-y_1)e^{\sigma_1^2}e^{\sigma_1\tilde{x} - \frac{\sigma_1^2}{2}}S_{t_1}^{(0)} - K^\delta\right)\phi(\tilde{x})\, \mathrm{d}\tilde{x}\right], \quad \text{where } \tilde{x} := x - \sigma_1 \\
&= -\partial_k \mathbb{E}\left[\int h\left((1-y_1)e^{\sigma_1^2}e^{\sigma_1\tilde{x} - \frac{\sigma_1^2}{2}}S_{t_1}^{(0)} - k\right)\phi(\tilde{x})\, \mathrm{d}\tilde{x}\right]\Big|_{k=K^\delta}, \qquad \text{and } e^{\sigma_1\tilde{x} - \frac{\sigma_1^2}{2}} = \frac{S_T^{(0)}}{S_{t_1}^{(0)}} \\
&= -\partial_k \mathbb{E}\left[h\left((1-y_1)e^{\sigma^2(T-t_1)}S_T^{(0)} - k\right)\right]\Big|_{k=K+\delta_1}.
\end{aligned}
$$

5. With the first order Taylor expansion,

$$
\begin{aligned}
h(S_T - K) &\approx h(\bar{S}_T^{(0)} - K) + (S_T - \bar{S}_T^{(0)})h'(\bar{S}_T^{(0)} - K) \\
&\approx h(\bar{S}_T^{(0)} - K) + \delta_1 h'(\bar{S}_T^{(0)} - K) - \delta_1\frac{S_T^{(0)}}{S_{t_1}^{(0)}}h'(\bar{S}_T^{(0)} - K)
\end{aligned}
$$

By taking the expectations we find the three terms calculated previously. □