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Introduction to diffusion process

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December 12, 2017

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1 Introduction

This introduction deliberately is incomplete from a mathematical point of vue. Its aim is just to introduce some notions on stochastic calculus without being rigorous at all.

1.1 Brownian motion

Let us consider a set of r.v. in $\mathbb{R}$, X_t is the value at time t. Between the times t and $t+h$ with $h>0$,

$$\Delta X_t^h \;:=\; X_{t+h} - X_t \tag{1.1}$$
$$= H_t\, \varepsilon_t^h. \tag{1.2}$$

This is the fundamental hypothesis, when h is small ΔX_t^h is the product of H_t which is a continuous function observable at time t, and ε_t^h r.v. independent from all that happen until t and s.t. its law only depends on h.

Remark. This assumption is quite natural if X_t is deterministic then with H_t is its derivative and $\varepsilon_t^h = h$; and you find the Taylor expansion.

Let's take $h = h_1 + h_2$ with $h_1, h_2 > 0$, then

$$X_{t+h} = X_{t+h_1} + H_{t+h_1}\varepsilon_{t+h_1}^{h_2} \tag{1.3}$$
$$= X_t + \underbrace{H_t\varepsilon_t^{h_1} + H_{t+h_1}\varepsilon_{t+h_1}^{h_2}}_{H_t\varepsilon_t^h}. \tag{1.4}$$

And with $h_1 < h$ very small by continuity we have $H_{t+h_1} \approx H_t$, so we deduce

$$\varepsilon_t^h \approx \varepsilon_t^{h_1} + \varepsilon_{t+h_1}^{h_2}. \tag{1.5}$$

By iterating,

$$\varepsilon_t^h \approx \varepsilon_t^{\frac{h}{n}} + \varepsilon_{t+\frac{h}{n}}^{\frac{h}{n}} + \cdots + \varepsilon_{t+\frac{(n-1)h}{n}}^{\frac{h}{n}}, \tag{1.6}$$

so ε_t^h is the sum of n r.v. of same law, by the central limit ε_t^h follows a normal law. We note m_h its mean and σ_h^2 its variance. With (1.5) we deduce

$$\begin{cases} m_{h_1+h_2} = m_{h_1} + m_{h_2} \\ \sigma^2_{h_1+h_2} = \sigma^2_{h_1} + \sigma^2_{h_2} \end{cases} \tag{1.7}$$
$$\Leftrightarrow \begin{cases} m_h = bh \\ \sigma_h^2 = \sigma^2 h \end{cases}, \quad b\in\mathbb{R},\ \sigma \geq 0 \tag{1.8}$$
$$\Leftrightarrow \varepsilon_t^h = bh + \sigma(B_{t+h} - B_t). \tag{1.9}$$

where $(B_t, t\geq 0)$ is a brownian motion.

Definition 1.1 (Brownian motion)**.** Continuous set of r.v. where each r.v. is gaussian, s.t. $B_t \sim \mathcal{N}(0,t)$ and $B_{t+h} - B_t$ is independent from $(B_s, s\in[0,t])$.

We then have with this definition,

$$X_{t+h} = X_t + H_t(bh + \sigma(B_{t+h} - B_t)). \tag{1.10}$$

If $\sigma = 0$, we simply have the differential equation $\frac{\mathrm{d}X_t}{\mathrm{d}t} = bH_t$, *i.e.*

$$X_t = X_0 + \int_0^t bH_s\, \mathrm{d}s. \tag{1.11}$$

Else if $\sigma > 0$,

$$X_t = X_0 + \int_0^t bH_s\,\mathrm{d}s + \underbrace{\int_0^t \sigma H_s\,\mathrm{d}B_s}_{\text{stochastic integral}}. \tag{1.12}$$

1.2 Stochastic integral

Here $(H_t, t\geq 0)$ is a continuous process, we also assume that for all t, H_t is observable. And it is very important that $(B_{t+h} - B_t, h > 0)$ and $(H_s, B_s, s\in[0,t])$ are independent.

We now want to give a sense to the stochastic integral $\int_0^t H_s\,\mathrm{d}B_s$. Let us begin with the simple case where the function $t\mapsto H_t$ is a floor function, *i.e.*

$$H_t := \sum_{i=1}^n H_{t_i}\mathbb{1}_{[t_i,t_{i+1}[}(t).$$

We then define its stochastic integral,

$$\int_0^t H_s\,\mathrm{d}B_s \triangleq \sum_{i=1}^n H_{t_i}(B_{t_{i+1}\wedge t} - B_{t_i\wedge t}). \tag{1.13}$$

Lemma 1.1. *Let ξ_i and η_i, $i\in[\![1,n]\!]$, square integrable r.v. s.t. for all i, $\mathbb{E}[\eta_i] = 0$ and η_i is independent from $(\xi_j,\eta_k, j\in[\![1,i]\!], k\in[\![1,i[\![)$. Then,*

(i) $\mathbb{E}\left[\sum_{i=1}^n \xi_i\eta_i\right] = 0$;

(ii) $\mathbb{E}\left[\left(\sum_{i=1}^n \xi_i\eta_i\right)^2\right] = \sum_{i=1}^n \mathbb{E}[\xi_i^2]\mathbb{E}[\eta_i^2]$.

Proof. For (i).

$$\mathbb{E}\left[\sum_{i=1}^n \xi_i\eta_i\right] = \sum_{i=1}^n \mathbb{E}[\xi_i\eta_i], \qquad \text{(linearity)}$$
$$= \sum_{i=1}^n \mathbb{E}[\xi_i]\underbrace{\mathbb{E}[\eta_i]}_{=0}, \qquad \xi_i \perp\!\!\!\perp \eta_i$$

For (ii).

$$\mathbb{E}\left[\left(\sum_{i=1}^n \xi_i\eta_i\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^n (\xi_i\eta_i)^2 + 2\sum_{1\leq i<j\leq n}\xi_i\xi_j\eta_i\eta_j\right]$$
$$= \sum_{i=1}^n \mathbb{E}[\xi_i^2]\mathbb{E}[\eta_i^2] + 2\sum_{1\leq i<j\leq n}\mathbb{E}[\xi_i\xi_j\eta_i]\underbrace{\mathbb{E}[\eta_j]}_{=0}.$$

□

Lemma 1.2. *Let $(H_t, t \geq 0)$ be a constant piecewise process. If $\mathbb{E}[H_s^2] < \infty$, then for all $t \geq 0$,*

(i) $\mathbb{E}\left[\int_0^t H_s \, \mathrm{d}B_s\right] = 0$;

(ii) $\mathbb{E}\left[\left(\int_0^t H_s \, \mathrm{d}B_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 \, \mathrm{d}s\right]$.

Proof. Write the integral with the definition (1.13) then take $\xi_i := H_{t_i}$ and $\eta_i := B_{t_{i+1}\wedge t} - B_{t_i \wedge t}$ from Lemma 1.1. □

We can then extend the stochastic integral to other process as each process can be written as the limit of constant piecewise processes.

1.3 Girsanov theorem

Proposition 1.3. *Let $r > 0$ and $f \in L^2([0,r], \mathrm{d}s)$, then for all $t \in [0,r]$, $\int_0^t f(s) \, \mathrm{d}Bs \sim \mathcal{N}\left(0, \int_0^t f^2(s) \, \mathrm{d}s\right)$.*

Proof. $\exists (f_n)$, $f_n := \sum_{i=0}^{p_n-1} \alpha_i^n \mathbb{1}_{[t_{i+1}^n, t_i^n[}$ and $f_n \to f$ in $L^2([0,r], \mathrm{d}s)$. We saw that $\int_0^t f_n(s) \, \mathrm{d}B_s \to \int_0^t f(s) \, \mathrm{d}B_s$ in $L^2(\Omega)$. And $\int_0^t f_n(s) \, \mathrm{d}B_s = \sum_{i=1}^{p_n} \alpha_i^n (B_{t_{i+1}^n \wedge t} - B_{t_i^n \wedge t})$ follows the normal law $\mathcal{N}\left(0, \int_0^t f_n(s)^2 \, \mathrm{d}s\right)$. □

Let us note

$$Z \quad := \quad \exp\left(\int_0^r f(s) \, \mathrm{d}B_s - \frac{1}{2}\int_0^r f(s)^2 \, \mathrm{d}s\right).$$

And let us define the probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by,

$$\mathbb{Q} = \mathbb{E}[Z \mathbb{1}_A] = \int_\Omega \mathbb{1}_A Z \, \mathrm{d}\mathbb{P} \ , \quad A \in \mathcal{F}.$$

Then for all r.v. X,

$$\mathbb{E}_{\mathbb{Q}}[X] = \int_\Omega X \, \mathrm{d}\mathbb{Q} = \int_\Omega XZ \, \mathrm{d}\mathbb{P} = \mathbb{E}[XZ].$$

Theorem 1.4 (Girsanov's theorem)**.** *The process defined by*

$$\widetilde{B}_t \quad := \quad B_t \; - \; \int_0^t f(s) \, \mathrm{d}s,$$

is a brownian motion on $\mathbb{Q}$.

1.4 Itô's formula

Theorem 1.5. *Let $t > 0$ and $0 =: t_0^n < \cdots < t_{p_n}^n := t$ a set of subdivisions of $[0,t]$ where the interval $\lim_{n\to\infty} \max_{0 \leq i \leq p_n - 1} \left(t_{i+1}^n - t_i^n\right) = 0$. Then,*

$$\sum_{i=0}^{p_n-1} \left(B_{t_{i+1}^n} - B_{t_i^n}\right)^2 \xrightarrow[n\to\infty]{L^2} t$$

Proof. We just have to see that for all $f : \mathbb{R} \to \mathbb{R}$ bounded measurable function,

$$\Sigma_n := \sum_{i=0}^{p_n-1} f\left(B_{t_i}^n\right)\left(\left(B_{t_{i+1}^n} - B_{t_i^n}\right)^2 - \left(t_{i+1}^n - t_i^n\right)\right) \xrightarrow{L^2} 0$$

We will use the Lemma 1.1 with $\xi_i := f\left(B_{t_i^n}\right)$ and $\eta_i := (B_{t_{i+1}^n} - B_{t_i^n})^2 - (t_{i+1}^n - t_i^n)$.

$$\mathbb{E}\left[\Sigma_n^2\right] \quad = \quad \sum_{i=0}^{p_n-1} \mathbb{E}\left[f^2(B_{t_i^n})\right] \mathbb{E}\left[\eta_i^2\right] \ ,$$

and η_i has the same law as $(t_{i+1}^n - t_i^n)(B_1^2 - 1)$ and $\mathbb{E}[(B_1^2 - 1)^2] = 2$ as $(B_1^2 - 1) \sim \chi^2(1)$. We then note $c := \sup_{x\in\mathbb{R}} 2f^2(x) < \infty$,

$$\begin{aligned}
\mathbb{E}\left[\Sigma_n^2\right] &\leq c \sum_{i=0}^{p_n-1} \left(t_{i+1}^n - t_i^n\right)^2 \\
&\leq c \max_{0\leq k \leq p_n - 1} \left(t_{k+1}^n - t_k^n\right) \sum_{i=0}^{p_n-1} \left(t_{i+1}^n - t_i^n\right) \\
&\leq c\, t \max_{0\leq k \leq p_n - 1} \left(t_{k+1}^n - t_k^n\right) \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

□

Theorem 1.6 (Itô's formula)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is in $\mathcal{C}^2$ then,*

$$f(B_t) \quad = \quad f(B_0) \; + \; \int_0^t f'(B_s) \, \mathrm{d}B_s \; + \; \frac{1}{2}\int_0^t f''(B_s) \, \mathrm{d}s.$$

Theorem 1.7 (Itô's formula)**.** *If $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is in $\mathcal{C}^{1,2}$ then,*

$$\begin{aligned}
f(t, B_t) \quad = \quad & f(0, B_0) \; + \; \int_0^t \frac{\partial f}{\partial t}(s, B_s) \, \mathrm{d}s \\
& + \; \int_0^t \frac{\partial f}{\partial x}(s, B_s) \, \mathrm{d}B_s \; + \; \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s) \, \mathrm{d}s.
\end{aligned}$$

2 Brownian motion

2.1 Definition and first properties

We work in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that is complete, *i.e.* that $\mathcal{F}$ contains all the $\mathbb{P}$-negligible sets. We call random processes all sets of random variables.

Definition 2.1 (Brownian motion)**.** $(B_t, t \geq 0)$ is a Brownian motion (real and null in 0) if:

(i) $t \mapsto B_t$ (the path) is a.s. continuous on $\mathbb{R}_+$.

(ii) $B_0 = 0$ a.s.

(iii) $\forall n \geq 2$, $\forall 0 \leq t_1 \leq \cdots \leq t_n$, $B_{t_n} - B_{t_{n-1}}, \ldots, B_{t_2} - B_{t_1}, B_{t_1}$ are independant.

(iv) $\forall t \geq s \geq 0$, $B_t - B_s \sim \mathcal{N}(0, t-s)$.

Remark. The Brownian motion is a process with independent increase (iii) and stationary (iv), *i.e.* it is a Levy process.

Definition 2.2 (Gaussian process)**.** $(X_t, t \in \mathbf{T})$ is a Gaussian process if for all $n \geq 1$ and for all $(t_1, \dots, t_n) \in \mathbf{T}^n$, $(X_{t_n}, \dots, X_{t_n})$ is a Gaussian vector.

Proposition 2.1. *$(X_t, t \geq 0)$ is a brownian motion iff a.s. $t \mapsto X_t$ is continuous on $\mathbb{R}_+$ and $(x_t, t \geq 0)$ is a centred gaussian process of covariance* $\mathrm{Cov}(X_t, X_s) = t \wedge s$, *$\forall s, t \geq 0$.*

Proof. "⇒". For all $0 \leq t_1 \leq \cdots \leq t_n$, $X_{t_n} - X_{t_{n-1}}, \dots, X_{t_2} - X_{t_1}, X_{t_1}$ are Gaussian and independant r.v. so $(X_{t_n} - X_{t_{n-1}}, \dots, X_{t_2} - X_{t_1}, X_{t_1})$ is a Gaussian vector, so $(X_1, \dots, X_n)$ is a centred Gaussian vector.
For all $t \geq s \geq 0$,

$$\begin{aligned} \mathrm{Cov}(X_t, X_s) &\triangleq \mathbb{E}[X_t X_s] - \underbrace{\mathbb{E}[X_t]\mathbb{E}[X_s]}_{=0} \\ &= \underbrace{\mathbb{E}[(X_t - X_s)]\mathbb{E}[X_s]}_{=0} + \mathbb{E}[X_s^2] \\ &= \mathbb{E}[(X_s - X_0)^2] = \mathrm{Var}[X_s - X_0] = s - 0. \end{aligned}$$

"⇐". (ii). $\mathrm{Var}[X_0] = \mathbb{E}[X_0^2] = 0$ and $\mathbb{E}[X_0] = 0$.
(iii). $\forall 0 \leq t_1 \leq \cdots \leq t_n$, $(X_{t_1}, \dots, X_{t_n})$ is a gaussian vector, so $(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$ is also a Gaussian vector. To show the independence we just have to show that the covariance is null. For all $1 \leq i < j \leq n$,

$$\begin{aligned} C &:= \mathrm{Cov}(X_{t_i} - X_{t_{i-1}}, X_{t_j} - X_{t_{j-1}}) \\ &= \mathrm{Cov}(X_{t_i} - X_{t_j}) - \mathrm{Cov}(X_{t_{i-1}} - X_{t_j} \\ &\quad - \mathrm{Cov}(X_{t_i} - X_{t_{j-1}} + \mathrm{Cov}(X_{t_{i-1}} - X_{t_{j-1}} \\ &= t_i - t_{i-1} - t_i + t_{i-1} = 0. \end{aligned}$$

(iv). For all $t \geq s \geq 0$, $X_t - X_s$ is a centred Gaussian r.v. and,

$$\begin{aligned} \mathrm{Var}[X_t - X_s] &= \mathrm{Var}[X_t] + \mathrm{Var}[X_s] - 2\mathrm{Cov}[X_t, X_s] \\ &= t + s - 2s = t - s. \end{aligned}$$

□

Theorem 2.2 (Wiener)**.** *The brownian motion exists.*

Proof. □

Proposition 2.3. *If B_t is a Brownian motion, then the following processes are also:*

(i) $X_t = -B_t$.

(ii) $X_t = \frac{1}{t} B_{\frac{1}{t}}$.

(iii) $a > 0$, $X_t = \frac{1}{\sqrt{a}} B_{at}$.

(iv) $s \geq 0$, $X_t = B_{t+s} - B_s$.

(v) $r > 0$, $X_t = B_r - B_{r-t}$, *avec* $t \in [0, r]$.

Example 2.1 (Brownian bridge)**.** Let B be a Brownian motion, we define $b_t := B_t - tB_t$ with $t \in [0,1]$. This is a centred Gaussian process with covariance $s \wedge t - st$.

- $(b_t, t \in [0,1]) \perp\!\!\!\perp B_1$.
- If $(b_t, t \in [0,1])$ is a Brownian bridge then $(b_{1-t}, t \in [0,1])$ also.
- If $(b_t, t \in [0,1])$ is a Brownian bridge then $B_t := (1+t) b_{\frac{t}{1+t}}$ is a Brownian motion.

Note that $b_t = (1-t) B_{\frac{t}{1-t}}$.

Example 2.2. We have $\lim_{t \underset{>}{\to} 0} B_t = 0$, a.s. By inverting time,

$$\lim_{t \to \infty} \frac{B_t}{t} = 0.$$

Theorem 2.4 (Lévy)**.** *Let $t > 0$ and $0 =: t_0^n < \cdots < t_{p_n}^n := t$ a set of subdivisions of $[0, t]$ where the interval* $\max_{0 \leq i \leq p_n - 1} \left(t_{i+1}^n - t_i^n \right) \xrightarrow[n \to \infty]{} 0$. *Then,*

$$\sum_{i=0}^{p_n - 1} \left(B_{t_{i+1}^n} - B_{t_i^n} \right)^2 \xrightarrow[n \to \infty]{L^2} t.$$

See the proof after Theorem 1.5.

2.2 Markov property

We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space. Define $(\mathcal{F}_t, t \geq 0)$ a filtration, *i.e.* a growing set of sub-tributes from $\mathcal{F}$. For all $0 \leq s \leq t$:

$$\mathcal{F}_0 \subset \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}.$$

Example 2.3. Let $(X_t, t \geq 0)$ be a stochastic process and define for all $t \geq 0$,

$$\mathcal{F}_t := \sigma(X_s, s \in [0, t]).$$

Then $(\mathcal{F}_t)$ is a filtration, and it is called the canonical filtration of the process X.

We say that $(B_t, t \geq 0)$ is a $(\mathcal{F}_t)$-Brownian motion if:

(i) $\forall t \geq 0$, B_t is $\mathcal{F}_t$-measurable (adapted) ;

(ii) $\forall s \geq 0$, $(B_{t+s} - B_s, t \geq 0)$ is a Brownian motion independent from $\mathcal{F}_s$.

Theorem 2.5 (Simple Markov property)**.** *We note B a Brownian motion and $(\mathcal{F}_t)$ its associated canonical filtration, then B is a $(\mathcal{F}_t)$-Brownian motion.*

In other words, for all $s \geq 0$, $(B_{t-s} - B_s, t \geq 0) \perp\!\!\!\perp \sigma(B_u, u \in [0, s])$.

Proof. We just have to show the independence, *i.e.* that the vectors $(B_{t_1+s} - B_s, \dots, B_{t_n+s} - B_s)$, $(B_{s_1}, \dots, B_{s_n})$ are independent for all $0 \leq t_1 < \cdots < t_n$, $0 \leq s_1 < \cdots < s_n \leq s$.

We have $\mathrm{Cov}(B_{t_i+s} - B_s, B_{s_j}) = \mathrm{Cov}(B_{t_i+s}, B_{s_j}) - \mathrm{Cov}(B_s, B_{s_j}) = s_j - s_j = 0$. And $(B_{t_1+s} - B_s, \dots, B_{t_n+s} - B_s, B_{s_1}, \dots, B_{s_n})$ is a gaussian vector, so the two previous vectors are independent. □

Let $(\mathcal{F}_t)$ be a filtration. For all $t \geq 0$ we note

$$\mathcal{F}_{t+} \quad := \quad \bigcap_{u>t} \mathcal{F}_u.$$

It is clear that it is also a filtration.

Theorem 2.6 (The 0–1 law of Blumenthal)**.** *Let B be a Brownian motion and $(\mathcal{F}_t)$ its associated canonical filtration. Then for all $A \in \mathcal{F}_{0+}$, $\mathbb{P}(A) = 0$ or 1.*

Proof. Let us first show that B is a $(\mathcal{F}_{t+})$-Brownian motion. As previously we just have to show the independence (the other points are clear). So we have to show that, for $A \in \mathcal{F}_{s+}$, $0 \leq t_1 < \cdots < t_n$ and $F : \mathbb{R}^n \to \mathbb{R}$ bounded and continuous:

$$\begin{aligned}\mathbb{E}[\mathbb{1}_A F(B_{t_1+s} - B_s, \ldots, B_{t_n+s} - B_s)] \quad &= \\ \mathbb{P}(A) E[\mathbb{1}_A F(B_{t_1+s}, \ldots, B_{t_n+s})].\end{aligned} \tag{2.1}$$

Let $\varepsilon > 0$, the process $t \mapsto B_{t+s+\varepsilon} - B_{s+\varepsilon}$ is independent from $\mathcal{F}_{s+\varepsilon}$ and *a fortiori* $\mathcal{F}_{s+}$. So,

$$\begin{aligned}\mathbb{E}[\mathbb{1}_A F(B_{t_1+s+\varepsilon} - B_{s+\varepsilon}, \ldots, B_{t_n+s+\varepsilon} - B_{s+\varepsilon})] \quad &= \\ \mathbb{P}(A) E[\mathbb{1}_A F(B_{t_1+s+\varepsilon}, \ldots, B_{t_n+s+\varepsilon})].\end{aligned}$$

By $\varepsilon \to 0$, an argument of continuity and the dominated convergence theorem we obtain (2.1).

So we have that $\mathcal{F}_{0+}$ is independent from $\sigma(B_t, t \geq 0)$. If $A \in \mathcal{F}_{0+} = \bigcap_{u>0} \mathcal{F}_u \subset \sigma(B_t, t \geq 0)$, then $A \perp\!\!\!\perp A$ and $\mathbb{P}\{A\} = \mathbb{P}\{A\}^2$. □

Example 2.4. Let $\tau := \inf\{t > 0 : B_t > 0\}$, then $\tau = 0$ a.s.

Indeed,

$$\{\tau = 0\} \quad = \quad \bigcap_{\substack{t \in]0,\varepsilon] \\ t \in \mathbb{Q}}} \{ \sup_{\substack{s \in [0,t] \\ s \in \mathbb{Q}}} B_s > 0\} \ , \quad \forall \varepsilon > 0$$

So $\{\tau = 0\} = \bigcap_{\varepsilon > 0} \mathcal{F}_\varepsilon = \mathcal{F}_{0+}$, we now know that $\mathbb{P}\{\tau = 0\} = 0$ or 1. For all $t \geq 0$, $\mathbb{P}\{\tau \leq t\} \geq \mathbb{P}\{B_t > 0\} = \frac{1}{2}$ and $\mathbb{P}\{\tau \leq t\} = \lim_{\varepsilon \to 0} \downarrow \mathbb{P}\{\tau \leq t\} \geq \frac{1}{2}$, then $\tau = 0$ a.s.

By time inversion we also see that $\{t > 0 : B_t = 0\}$ is non bounded a.s.

Let $(\mathcal{F}_t)$ be a filtration, we define

$$\mathcal{F}_\infty \quad := \quad \sigma(\mathcal{F}_t, t \geq 0) \ ,$$

the tiniest tribe that contains all the elements of the tribes $\mathcal{F}_t$.

Definition 2.3 (Stopping time)**.** The application $T : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ is a stopping time if for all $t \geq 0$, $\{T \leq t\} \in \mathcal{F}_t$.

Example 2.5. The constant $T = t$ is a stopping time. And $T_a := \inf\{t > 0 : B_t = a\}$ is also a stopping time, indeed $\{T_a \leq t\} = \{\sup_{0 \leq s \leq t} B_s = a\} \in \mathcal{F}_t$.

Definition 2.4. Let T be a stopping time. The tribe of previous events to T is

$$\mathcal{F}_T \quad := \quad \{A \in \mathcal{F}_\infty : \forall t \geq 0, \cap\{T \leq t\} \in \mathcal{F}_t\}.$$

Theorem 2.7 (Strong Markov property)**.** *We note B a Brownian motion and $(\mathcal{F}_t)$ its associated canonical filtration. Let T be a stopping time. Conditionaly to $\{T < \infty\}$, the process $(B_{T+t} - B_T, t \geq 0)$ is a Brownian motion independent from $(\mathcal{F}_T)$.*

Proof. □

Theorem 2.8 (Reflexivity)**.** *Let B be a Brownian motion. With $t \geq 0$, we note $S_t := \sup_{s \in [0,t]} B_s$. Then,*

$$\mathbb{P}\{S_t \geq a, B_t \leq b\} \quad = \quad \mathbb{P}\{B_t \geq 2a - b\}.$$

for all $a \geq 0$, $b \leq a$.

In particular for all fixed t, $S_t \sim |B_t|$.

Remark. It is just for t fixed, the equality in law between $(S_t, t \geq 0)$ and $(|B_t|, t \geq 0)$ is false.

Proof.

$$\begin{aligned}\mathbb{P}\{S_a \geq t, B_t \leq b\} \quad &= \quad \mathbb{P}\{T_a \leq t, B_t \leq t\} \\ &= \quad \mathbb{P}\{T_a \leq t, \tilde{B}_{t-T_a} \leq b - a\} \ , \\ &\qquad \tilde{B}_s := B_{s+T_a} - B_{T_a} \\ &= \quad \mathbb{P}\{(T_a, \tilde{B}) \in A_t\} \ ,\end{aligned}$$

where $A_t := \{(u, x) \in \mathbb{R}_+ \times ?, 0 \leq u \leq t, x(t-u) \leq b - a)\}$. We have $P_{(T_a, \tilde{B})} = P_{T_a} \otimes P_{\tilde{B}}$ as $T_a \perp\!\!\!\perp \tilde{B}$. And $\tilde{B} \sim -\tilde{B}$, so,

$$\begin{aligned}\mathbb{P}\{S_a \geq t, B_t \leq b\} \quad &= \quad \mathbb{P}\{(T_a, -\tilde{B}_t) \in A_t\} \\ &= \quad \mathbb{P}\{T_a \leq t, -\tilde{B}_{t-T_a} \leq b - a\} \\ &= \quad \mathbb{P}\{\underbrace{T_a \leq t}_{S_t \geq a}, \underbrace{-(B_t - a) \leq b - a}_{B_t \geq 2a - b}\} \\ &= \quad \mathbb{P}\{B_t \geq 2a - b\} \ , \quad \text{as } 2a - b \geq a.\end{aligned}$$

To prove that they have the same law,

$$\begin{aligned}\mathbb{P}\{S_a \geq a\} \quad &= \quad \mathbb{P}\{S_a \geq a, B_t \leq a\} + \mathbb{P}\{S_a \geq a, B_t > a\} \\ &= \quad \mathbb{P}\{B_t \geq 2a - a\} + \mathbb{P}\{\underbrace{B_t}_{= -B_t} \geq a\} \\ &= \quad \mathbb{P}\{|B_t| \geq a\}.\end{aligned}$$

□

Corollary. *For all $t > 0$, the density of (S_t, B_t) is*

$$\frac{2(2a-b)}{\sqrt{2at^3}} \exp\left(-\frac{(2a-b)^2}{2t}\right) \mathbb{1}_{\{a>0, b<a\}}.$$

Example 2.6. Let $t > 0$ and $a > 0$,

$$\begin{aligned}\mathbb{P}\{T_a \leq t\} \quad &= \quad \mathbb{P}\{S_t \geq a\} \\ &= \quad \mathbb{P}\{|B_t| \geq a\} \\ &= \quad \mathbb{P}\{|\sqrt{t} B_1| \geq a\} \\ &= \quad \mathbb{P}\left\{t \geq \frac{a^2}{|B_1|^2}\right\}.\end{aligned}$$

So $T_a \sim \frac{a^2}{B_1^2}$

2.3 Semi-group of the Brownian motion

Let $s \geq 0$ and $f : \mathbb{R} \to \mathbb{R}_+$ measurable, we can write

$$\begin{aligned} \mathbb{E}[f(B_{t+s})|\mathcal{F}_s] &= \mathbb{E}[f(B_{t+s} - B_s + B_s)|\mathcal{F}_s] \\ &= (P_t f)(B_s) \ , \end{aligned}$$

where for all $x \in \mathbb{R}$, $P_t f := \mathbb{E}[f(B_t + x)] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) f(y)\mathrm{d}y$.

$(P_t, t \geq 0)$ is the semi-group of B. Indeed,

$$\begin{aligned} P_{t+s}f(x) &\triangleq \mathbb{E}[f(B_{t+s} + x)] \\ &= \mathbb{E}\left[\mathbb{E}[f(B_{t+s} + x)|\mathcal{F}_t]\right] \\ &= \mathbb{E}[(P_s f)(B_t + x)] \\ &= P_t(P_s f)(x). \end{aligned}$$

Proposition 2.9 (Feller property)**.** *Let $f \in \mathcal{C}_0$ (continuous s.t. $\lim_{|x|\to\infty} f(x) = 0$), then $P_t f \in \mathcal{C}_0$ and $\lim_{t\downarrow 0} P_t f = f$ uniform on $\mathbb{R}$.*

Proposition 2.10 (Infinitesimal generator)**.** *If $f \in \mathcal{C}_c^2$ (class $\mathcal{C}^2$ on a compact space), then $\lim_{x\downarrow 0} \frac{P_t f(x) - f(x)}{t} = \frac{1}{2} f''(x)$.*

There is a strong link with the heat equation. Let $u(t,x) := P_t f(x)$. On a $u(0,x) = f(x)$. If f is measurable and bounded then,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}.$$

3 Continuous-time martingale

3.1 Progressive processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered (probability) space. We define

$$\mathcal{F}_{t+} = \bigcap_{u>y} \mathcal{F}_u \ ,$$

for $t \geq 0$.

Definition 3.1 (Right continuity)**.** We say that $(\mathcal{F}_t)_{t\geq 0}$ is right continuous if $\forall t > 0$, $\mathcal{F}_{t+} = \mathcal{F}$.

Definition 3.2 (Complete)**.** We say that $(\mathcal{F}_t)_{t\geq 0}$ is complete if $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets.

Proposition 3.1. *We have a bunch of properties about the process $(X_t, t \geq 0)$:*

(i) it is right (respectively left) continuous if a.s. $t \mapsto X_t$ is right (respectively left) continuous.

(ii) it is adapted if $\forall t \geq$, X_t is $\mathcal{F}_t$-measurable ;

(iii) it is progressive (or progressively measurable) if $\forall t \geq 0$, $(s,\omega) \mapsto X_s(\omega)$ is measurable along $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$.

We have $(X_t, t \geq 0)$ progressive $\Rightarrow$ $(X_t, t \geq 0)$ adapted. The reverse is generally false.

Proposition 3.2. *If $(\mathcal{F}_t)$ is complete and $(X_t, t \geq 0)$ a process in $\mathbb{R}^d$, adapted, right (or left) continuous, then $(X_t, t \geq 0)$ is progressive.*

Proof. Let $(X_t, t \geq 0)$ be right continuous, so $\forall t \geq 0$, $\forall n \geq 1$ we set

$$X_s^{(n)} := X_{\frac{\left(\lfloor \frac{ns}{t} \rfloor + 1\right)t}{n} \wedge t} \ ,$$

with $s \in [0,t]$. And $\forall s \in [0,t]$, $X_s^{(n)} \to X_s$. Now for all $A \in \mathcal{B}(\mathbb{R}^d)$, $\{(s,\omega) \in [0,t] \times \Omega : X_s^{(n)}(\omega) \in A\}$ can be written as

$$\bigcup_{k=1}^n \left(\left[\frac{(k-1)t}{n}, \frac{kt}{n}\right[\times X_{\frac{kt}{n}}^{-1}(A)\right) \bigcup \left(\{t\} \times X_t^{-1}(A)\right).$$

Where $X_{\frac{kt}{n}}^{-1}(A)$ is $\mathcal{F}_{\frac{kt}{n}}$-measurable so $\mathcal{F}_t$-measurable. And $\mathcal{F}_t$ is complete so $\mathbb{P}$-null sets make no trouble and X_s is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable as limit of $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable functions. □

Let $A \subset \mathbb{R}_+ \times \Omega$ and $A \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$, is progressive if $(x_t(\omega) := \mathbb{1}_{\{(t,\omega)\in A}, t \geq 0)$ is a progressive process. And

- $\{A$ progressive$\}$ is a tribe called the progressive tribe ;
- $(X_t, t \geq 0)$ is pregressive $\Leftrightarrow$ $(t,\omega) \mapsto X_t(\omega)$ along the progressive tribe.

3.2 Stopping time

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered space. We recall

$$\mathcal{F}_\infty := \sigma(\mathcal{F}_t, t \geq 0) \ ,$$

and that $T : \Omega \to \mathbb{R} \cup \{\infty\}$ is a stopping time if $\forall t \geq 0$, $\{T \leq t\} \in \mathcal{F}_t$.

Definition 3.3. If T is a stopping a time we set

$$\mathcal{F}_T := \{A \in \mathcal{F}_\infty : A \cap \{T \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}.$$

Proposition 3.3. *We have the following properties:*

(i) if $S \leq T$ stopping times $\Rightarrow$ $\mathcal{F}_S \subset \mathcal{F}_T$;

(ii) S, T stooping times $\Rightarrow$ $S \vee T$, $S \wedge T$ stopping times and $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$;

(iii) $\{S \leq T\}, \{S < T\}, \{S = T\} \in \mathcal{F}_{S\wedge T}$;

(iv) $(\mathcal{F}_t)$ right continuous and T stopping time $\Leftrightarrow$ $\{T < t\} \in \mathcal{F}_t$, $\forall t > 0$;

(v) $(\mathcal{F}_t)$ right continuous and (T_n) set of stopping time $\Rightarrow$ $T := \inf_{n\geq 1} T_n$ stopping time and $\mathcal{F}_T = \bigcap_{n\geq 1} \mathcal{F}_{T_n}$.

Proof. (i). $\forall A \in \mathcal{F}_S$ we have $A \in \mathcal{F}_\infty$ this is trivial. We just have to show that $A \cap \{T \leq t\} \in \mathcal{F}_t$ for all t. Indeed,

$$A \cap \{T \leq t\} \in \mathcal{F}_t \quad = \quad \underbrace{A \cap \{S \leq t\}}_{\in \mathcal{F}_t} \cap \underbrace{\{T \leq t\}}_{\in \mathcal{F}_t}.$$

(ii). For all t,

$$\{S \wedge T\} \quad = \quad \underbrace{\{S \leq t\}}_{\in \mathcal{F}_t} \cap \underbrace{\{T \leq t\}}_{\in \mathcal{F}_t},$$

so $\{S \wedge T\} \in \mathcal{F}_t$ and it's a stopping time. We use the same proof for $S \vee T$.
$\mathcal{F}_{S\wedge T} \subset \mathcal{F}_S$ and $\subset \mathcal{F}_T$ so $\mathcal{F}_{S\wedge T} \subset \mathcal{F}_S \cap \mathcal{F}_T$. Now let $A \in \mathcal{F}_S \cap \mathcal{F}_T$, and for all t,

$$A \cap \{S \wedge T \leq t\} \quad = \quad \underbrace{(A \cap \{S \leq t\})}_{\in \mathcal{F}_t} \cup \underbrace{(A \cap \{T \leq t\})}_{\in \mathcal{F}_t} \in \mathcal{F}_t ,$$

so $A \in \mathcal{F}_{S\wedge T}$ and $\mathcal{F}_{S\wedge T} \supset \mathcal{F}_S \cap \mathcal{F}_T$. □

Proposition 3.4. *Let T be a stopping time, for all $n \geq 0$ set*

$$T_n \quad := \quad \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbb{1}_{\left\{\frac{k-1}{2^n} < T \leq \frac{k}{2^n}\right\}} \; + \; \infty \mathbb{1}_{\{\infty\}} ,$$

then $(T_n, n \geq 1)$ is a stopping time set that converges towards T.

Proof. For all t,

$$\{T_n \leq t\} \quad = \quad \{T < t\} \cap \underbrace{\{T_n \leq t\}}_{\in \mathcal{F}_T} \in \mathcal{F}_t.$$

So it's a stopping time. □

Theorem 3.5. *Let $(X_t, t \geq 0)$ be a progressive process in $\mathbb{R}^d$ and T a stopping time. Then $\mathbb{1}_{\{T<\infty\}} X_T$ is $\mathcal{F}_T$-measurable.*

If furthermore $X_t(\omega) \xrightarrow[t\to\infty]{} X_\infty(\omega) \in \mathbb{R}^d$, $\forall \omega \in \Omega$ then X_T is $\mathcal{F}_T$-measurable.

3.3 Continuous time martingale

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered space.

Definition 3.4 (Martingale)**.** We call $(M_t, t \geq 0)$ a martingale (respectively submartingale, supermartingale) if

(i) $(M_t, t \geq 0)$ is adapted ;

(ii) $\forall t \geq 0$, $\mathbb{E}[|M_t|] < \infty$;

(iii) $\forall t \geq s \geq 0$, $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ a.s. (respectively $\geq$, $\leq$).

Remark. It is clear that for $(M_t, t \geq 0)$ submartingale (resp. supermartingale), $t \mapsto \mathbb{E}[M_t]$ is growing (resp. is decreasing).

Example 3.1. Let $B = (B_t, t \geq 0)$ be a $(\mathcal{F}_t)$-brownian motion, then the following processes are martingales

(i) $(B_t, t \geq 0)$;

(ii) $(B_t^2 - 1, t \geq 0)$;

(iii) $(e^{\theta B_t - \frac{\theta^2}{2} t}$, with $\theta \in \mathbb{R}$.

Remark. If $(M_t, t \geq 0)$ is a martingale and $f : \mathbb{R} \to \mathbb{R}$ convex s.t. $\mathbb{E}[|f(M_t)|] < \infty$, then $(f(M_t), t \geq 0)$ is a submartingale.

Theorem 3.6 (Maximal inequality)**.** *Let $(M_t, t \geq 0)$ be a submartingale right continuous, then*

$$\mathbb{P}\left\{\sup_{s\in[0,t]} M_s > \lambda\right\} \quad \leq \quad \frac{\mathbb{E}[|M_t|]}{\lambda} ,$$

for all $\lambda > 0$, $t \geq 0$.

Theorem 3.7 (Doob's inequality)**.** *Let $(M_t, t \geq 0)$ be a right continuous martingale and $p > 1 \in \mathbb{R}$. Then*

$$\left\| \sup_{s\in[0,t]} |M_s| \right\|_p \quad \leq \quad q\|M_t\|_p ,$$

for all $t \geq 0$, where $\frac{1}{p} + \frac{1}{q} = 1$.

And consequently

$$\left\| \sup_{s\geq 0} |M_s| \right\|_p \quad \leq \quad q \sup_{s\geq 0} \|M_s\|_p$$

3.4 Convergence and optimal stopping theorem

Theorem 3.8. *Let $(M_t, t \geq 0)$ be right continuous submartingale s.t. $\sup_{t\geq 0} \mathbb{E}[M_t] < \infty$ (one can show that a equivalent is $\sup_{t\geq 0} \mathbb{E}[|M_t|] < \infty$) then,*

$$M_\infty \quad := \quad \lim_{t\to\infty} M_t \quad \text{exists a.s.}$$

and $\mathbb{E}[|M_\infty|] < \infty$.

Proof. We define $D \subset \mathbb{R}_+$ a countable dense space. With $a < b \in \mathbb{R}$, $N_{ab}([0, t] \cap D)$ is the number of grows of $(M_s, s \in [0, t] \cap D)$ along $[a, b]$. We have

$$\begin{aligned} \mathbb{E}[N_{ab}([0, t] \cap D)] \quad &\leq \quad \frac{\mathbb{E}[(M_t - a)^+]}{b - a} \\ &\leq \quad \frac{\sup_{u\geq 0} \mathbb{E}[(M_u)^+ + |a|]}{b - a} \; < \; \infty. \end{aligned}$$

Then with $t \to \infty$ we have for all $a < b$, $N_{ab}(D) < \infty$ a.s. and so $\lim_{\substack{t\to\infty \\ t\in D}} M_t$ exists.

Then with Fatou we verify that this limit is not $\pm\infty$, and finally as D is dense we have the result on $\mathbb{R}_+$ by writing the definition of the limit. □

Corollary. *Let $(M_t, t \geq 0)$ be a positive right continuous spermartingale, then $M_t \xrightarrow[t\to\infty]{a.s.} M_\infty$ and $\mathbb{E}[M_\infty] \leq \mathbb{E}[M_0] < \infty$.*

Theorem 3.9. *Let $p > 1$ be a real number and $(M_t, t \geq 0)$ a right continuous martingale s.t.* $\sup_{t\geq 0} \mathbb{E}[|M_t|^p] < \infty$*, then*

$$M_t \quad \xrightarrow[t\to\infty]{L^P \ a.s.} \quad M_\infty .$$

Proof. □

Theorem 3.10. *Let $(M_t, t \geq 0)$ be a right continuous submartingale uniformly integrable (UI), then*

(i) $M_t \xrightarrow[t\to\infty]{L^1} m_\infty$ *;*

(ii) $M_t \xrightarrow[t\to\infty]{a.s.} m_\infty$ *;*

(iii) $\forall t \geq 0$, $M_t \leq \mathbb{E}[M_\infty|\mathcal{F}_t]$ *a.s.*

Theorem 3.11 (Optimal stopping theorem)**.** *Let $(M_t, t \geq 0)$ be a right continuous submartingale and $S \leq T$ stopping times. If (a) $(M_t, t \geq 0)$ UI or (b) $S \leq T$ bounded ($\exists C < \infty$ s.t. $T(\omega) \leq C$, for all $\omega \in \Omega$) then*

$$M_S \quad \leq \quad \mathbb{E}[M_t|\mathcal{F}_S] \quad a.s.$$

And consequently $\mathbb{E}[M_S] \leq \mathbb{E}[M_t]$.

Example 3.2. $(M_t, t \geq 0)$ a right continuous submartingale and $S \leq T$, bounded stopping times. Then $\mathbb{E}[M_T|\mathcal{F}_S] \geq M_{S\wedge T}$ a.s.

Indeed,

$$\begin{aligned}
\mathbb{E}[M_T|\mathcal{F}_S] &= \mathbb{E}[M_T \mathbb{1}_{\{S\leq T\}}|\mathcal{F}_S] + \mathbb{E}[M_T \mathbb{1}_{\{S>T\}}|\mathcal{F}_S] \\
&= \mathbb{E}[M_{T\vee S} \mathbb{1}_{\{S\leq T\}}|\mathcal{F}_S] + \mathbb{E}[M_{T\wedge S} \mathbb{1}_{\{S>T\}}|\mathcal{F}_S] \\
&= \mathbb{1}_{\{S\leq T\}} \underbrace{\mathbb{E}[M_{T\vee S}|\mathcal{F}_S]}_{\geq M_S} + \mathbb{1}_{\{S>T\}} \underbrace{\mathbb{E}[M_{T\wedge S}|\mathcal{F}_S]}_{=M_{S\wedge T}} \\
&\geq M_{S\wedge T} .
\end{aligned}$$

Example 3.3. $(M_t, t \geq 0)$ a right continuous submartingale and T stopping time, then $(M_{T\wedge t}, t \geq 0)$ a right continuous submartingale.

Indeed for all $t \geq 0$, $M_{T\wedge t}$ is $(\mathcal{F}_{T\wedge t})$-measurable so $(\mathcal{F}_t)$-measurable. And for all $t \geq s \geq 0$, $\mathbb{E}[M_{T\wedge t}|\mathcal{F}_s] \geq M_{(T\wedge t)\wedge s} = M_{T\wedge s}$.

3.5 Example: Brownian motion

Example 3.4. Let $T_a := \inf\{t \geq 0 : B_t = a\}$. We know that $(M_t := e^{\theta B_t - \frac{\theta^2}{2} t}, t \geq 0)$ is a martingale. For $a > 0$, $(M_{T_a \wedge t}, t \geq 0)$ continuous bounded martingale, so UI. And so $\mathbb{E}[M_{T_a}] = \mathbb{E}[M_0] = 1$ with the optimal stopping theorem. On the other hand $\mathbb{E}[M_{T_a}] = \mathbb{E}[e^{\theta a - \frac{\theta^2}{2} T_a}]$ and then with $\lambda := \frac{\theta^2}{2}$,

$$\mathbb{E}[e^{-\lambda T_a}] \quad = \quad e^{-\sqrt{2\lambda a^2}} .$$

Example 3.5. Let (X_t, Y_t) be Brownian motion in $\mathbb{R}^2$ with $X_0 = 0$ and $Y_1 = 1$. We are looking for the distribution of X_T with $T := \inf\{t \geq 0 : Y_t = 0\}$.

Let $a \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\left[e^{iaX_T}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{iaX_T}|Y_t\right]\right] \\
&= \mathbb{E}\left[e^{-\frac{a^2}{2}T}\right] \\
&= e^{-|a|} ,
\end{aligned}$$

i.e. X_T has a standard Cauchy distribution.

4 Continuous semimartingales

4.1 Finite variation processes

Let $r > 0$ fixed and $a : [0, r] \to \mathbb{R}$ a finite variation continuous function with $a(0) = 0$. It is variation finite if $a = c_+ - c_-$ where $c_\pm : [0, r] \to \mathbb{R}$ growing functions.

We can assume that $c_\pm$ are continuous and $c_\pm(0) = 0$.

Let $\mu_\pm$ be Stieltjes measures associated to $c_\pm$, so

$$\mu_\pm([0, t]) \quad = \quad c_\pm(t) ,$$

for all $t \in [0, r]$.

Theorem 4.1 (Stieltjes)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ growing and right continuous function, then there exists a unique measure μ on $\mathbb{R}$ s.t.*

$$\mu(]a, b]) \quad = \quad F(b) - F(a) ,$$

for all $a < b \in \mathbb{R}$.

We can write

$$\mu \quad := \quad \mu_+ - \mu_- ,$$

where $\mu_\pm$ are signed measures on $[0, r]$. Actually this decomposition is unique and exists if $\mu_\pm$ are orthogonal, *i.e.* for all A measurable $\mu_+(A) = \mu_-(A^C) = 0$.

We can also write

$$|\mu| \quad := \quad \mu_+ + \mu_- ,$$

the total variation measure associated to the function a. And we have $\mu_\pm \ll |\mu|$ (*i.e.* $\forall A$ s.t. $|\mu|(A) = 0 \Rightarrow \mu_\pm(A) = 0$).

Proposition 4.2. *With the subdivisions $0 =: t_0 < \cdots < t_p := r$,*

$$|\mu|([0, r]) \quad = \quad \sup_{t_i} \sum_{i=1}^{p} |a(t_i) - a(t_{i-1})|.$$

Proof. □

We still have $a : [0, r] \to \mathbb{R}$ a finite variation continuous function with $a(0) = 0$, and $f : [0, r] \to \mathbb{R}$ measurable s.t. $\int_{[0,r]} |f| \, \mathrm{d}|\mu| < \infty$. Then we define

$$\begin{aligned}
\int_0^t f(x) \, \mathrm{d}a(s) &:= \int_0^t f(s) \, \mu(\mathrm{d}s) \\
&:= \int_0^t f(s) \, \mu_+(\mathrm{d}s) - \int_0^t f(s) \, \mu_-(\mathrm{d}s) \ \in \mathbb{R},
\end{aligned}$$

for all $t \in [0,r]$. And also

$$\int_0^t f(x)\,|\mathrm{d}a(s)| \;:=\; \int_0^t f(s)\,|\mu|(\mathrm{d}s)$$
$$:= \int_0^t f(s)\,\mu_+(\mathrm{d}s) + \int_0^t f(s)\,\mu_-(\mathrm{d}s) \;\in \mathbb{R}.$$

We have the triangular inequality

$$\left|\int_0^t f(s)\,\mathrm{d}a(s)\right| \;\leq\; \int_0^t |f(s)||\mathrm{d}a(s)|.$$

Lemma 4.3. *Let $f:[0,r]\to\mathbb{R}$ be continuous and a sequence of subdivisons of $[0,r]$: $0 =: t_0^n < \cdots < t_{p_n}^n := r$ from which the interval goes to 0. Then*

$$\int_0^t f(s)\,\mathrm{d}a(s) \;=\; \lim_{n\to\infty}\sum_{i=1}^{p_n} f(t_{i-1}^n)\left(a(t_i^n)-a(t_{i-1}^n)\right).$$

Proof. We define for all n, $f_n(s) := \sum_{i=1}^{p_n} f(t_{i-1}^n)\mathbb{1}_{]t_{i-1}^n,t_i^n]}(s)$, so

$$\sum_{i=1}^{p_n} f(t_{i-1}^n)\left(a(t_i^n)-a(t_{i-1}^n)\right) \;=\; \int_0^r f_n(s)\,\mathrm{d}a(s)$$
$$\xrightarrow[n\to\infty]{} \int_0^r f(s)\,\mathrm{d}a(s),$$

with dominated convergence. □

Now we can enlarge this result on $\mathbb{R}$. $a:\mathbb{R}_+\to\mathbb{R}$ is a finite variation on $\mathbb{R}_+$ (*i.e.* if for all $r>0$, a is a finite variation function on $[0,r]$) continuous function with $a(0)=0$. Let $f:\mathbb{R}_+\to\mathbb{R}$ measurable s.t. $\int_0^\infty |f(s)||\mathrm{d}a(s)| := \sup_{r>0}\int_0^r |f(s)||\mathrm{d}a(s)| < \infty$. Then

$$\int_0^\infty f(s)\,\mathrm{d}a(s) \;:=\; \lim_{r\to\infty}\int_0^r f(s)\,\mathrm{d}a(s).$$

Let us now define a filtered space $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$, with the filtration $(\mathcal{F}_t)$ that is right continuous and complete.

Proposition 4.4. *Let $(V_t,\ t\geq 0)$ be a continuous and adapted process, with finite variations and $V_0=0$. And $(H_t,\ t\geq 0)$ a progressive process, s.t. $\forall t>0$, $\int_0^t |H_s(\omega)||\mathrm{d}V_s(\omega)| < \infty$ a.s. Then,*

$$\left(\int_0^t H_s\,\mathrm{d}V_s,\ t\geq 0\right)$$

is a continuous, adapted process, null in 0 and with finite variations.

[1]Recall that if a is at finite variation then

$$|\mu|([0,r]) \;=\; \sup_{t_i}|a(t_i)-a(t_{i+1})|.$$

4.2 Continuous local martingales

We note with $X := (X_t,\ t\geq 0)$, T a stopping time,

$$X^T \;:=\; (X_{t\wedge T},\ t\geq 0).$$

Definition 4.1 (Local martingale (continuous))**.** We call $M := (M_t,\ t\geq 0)$ a local martingale if there exists $(T_n, n\geq 1)$ growing sequence of stopping time s.t. $T_n\uparrow\infty$ a.s. and that for all $n\geq 1$, $M^{T_n}-M_0$ is a continuous martingale UI.

But be careful we don't know anything about M_t, especially $\forall t$, $\mathbb{E}[|M_t|]\leq\infty$.

We say that (T_n) reduces M.

Remark.

- M if a continuous martingale $\Rightarrow M$ is a local continuous martingale (just take $T_n := n$).
- In the definition we can suppose that for all n, $M^{T_n}-M_0$ is a bounded martingale.
- M is a local continuous martingale, T stopping time, $\Rightarrow M^T$ local martingale.
- (T_n) reduces M, (S_n) a stopping time $\uparrow\infty$ a.s. $\Rightarrow$ $(T_n\wedge S_n)$ reduces M.
- A linear combination of two local martingale is a local martingale.

Proposition 4.5. *We have the following properties*

(i) M local positive continuous martingale, $\mathbb{E}[M_0]<\infty$ $\Rightarrow M$ supermartingale.

(ii) M local continuous martingale s.t. $\forall t\geq 0$, $\mathbb{E}[\sup_{s\in[0,t]}|M_s|]<\infty\Rightarrow M$ martingale.

(iii) M local continuous martingale s.t. $\mathbb{E}[\sup_{t\in[0,t]}|M_t|]<\infty\Rightarrow M$ martingale UI.

Proof. Let (T_n) reduce M.

(i). $\forall n$, $M^{T_n}-M_0$ a martingale UI. We have M_0 is $\mathcal{F}_0$-measurable so M^{T_n} is a martingale UI. And so for all $t\geq s\geq 0$,

$$\mathbb{E}[\underbrace{M_{T_n\wedge t}}_{\xrightarrow{a.s.}M_t}|\mathcal{F}_s] \;=\; \underbrace{M_{T_n\wedge s}}_{\xrightarrow{a.s.}M_s}.$$

So by Fatou's Lemma,

$$\mathbb{E}[M_t|\mathcal{F}_s] \;\leq\; \liminf_{n\to\infty}\mathbb{E}[M_{T_n\wedge t}|\mathcal{F}_s] \;=\; M_s.$$

(ii).$\forall t\geq s\geq 0$, $\forall A\in\mathcal{F}_s$, $\mathbb{E}[M_{T_n\wedge t}\mathbb{1}_A] = \mathbb{E}[M_{T_n\wedge s}\mathbb{1}_A]$. And $M_{T_n\wedge t}\to M_t$, so by dominated convergence $\mathbb{E}[M_{T_n\wedge t}\mathbb{1}_A]\to\mathbb{E}[M_t\mathbb{1}_A]$. We have the same for the right hand side in the equality. So for all $A\in\mathcal{F}_s$,

$$\mathbb{E}[M_t\mathbb{1}_A] \;=\; \mathbb{E}[M_s\mathbb{1}_A]$$
$$\Rightarrow\quad \mathbb{E}[M_t|\mathcal{F}_s] \;=\; \mathbb{E}[M_s|\mathcal{F}_s] \;=\; M_s.$$

□

Theorem 4.6. *Let M be a local continuous martingale, $M_0 = 0$ a.s. If M is at finite variation*[1]*, then*

$$\mathbb{P}\{M_t = 0,\ \forall t \geq 0\} \quad = \quad 1.$$

Proof. □

4.3 Quadratic variation

Theorem 4.7. *Let M be a local continuous martingale, then*

(i) There exists a unique[2] *continuous adapted growing process, null in 0, that we note $\langle M \rangle = (\langle M \rangle_t,\ t \geq 0)$ s.t. $\left(M_t^2 - \langle M \rangle_t,\ t \geq 0\right)$ is a local continuous martingale.*

(ii) For all $t > 0$, for all $0 =: t_0^n < \cdots < t_{p_n}^n := t$ sequence of subdivision of $[0, t]$ where the intervals go to 0,

$$\sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 \quad \xrightarrow[n\to\infty]{\mathbb{P}} \quad \langle M \rangle_t.$$

Proof. (Unicity). Assume that X and Y satisfy the conditions of $\langle M \rangle$, then $M^2 - X$, $M^2 - Y$ are local martingales $\Rightarrow$ $X - Y$, is a local martingale with finite variation, null in 0, and with Theorem 4.6, $\mathbb{P}\{X_t - Y_t = 0,\ t \geq 0\} = 1$. □

We call $\langle M \rangle$ the quadratic variation of M.

Example 4.1. Let B be a $(\mathcal{F}_t)$-brownian motion, then $\langle B \rangle_t = t$, consequence of the Levy theorem, or the fact that $(B_t^2 - t,\ t \geq 0)$ is a martingale.

Proposition 4.8. *Let M be a local continuous martingale and T a stopping time, then*

$$\langle M^T \rangle \quad = \quad \langle M \rangle^T.$$

Proof. We have by definition $(M^T)^2 - \langle M^T \rangle$ which is a local martingale, so let us just show that $(M^T)^2 - \langle M \rangle^T$ is also a local martingale and the proof is made. For $t \geq 0$,

$$(M^T)_t^2 - \langle M \rangle_t^T \quad = \quad M_{T\wedge t}^2 - \langle M \rangle_{T \wedge t},$$

which is a local martingale. □

Theorem 4.9. *Let M be a local continuous martingale, $M_0 = 0$,*

(i) $\mathbb{E}[\langle M \rangle_t] < \infty$, $\forall t \geq 0 \Leftrightarrow M$ is square-integrable. In this case $(M_t^2 - \langle M \rangle_t,\ t \geq 0)$ is a martingale null in 0.

(ii) $\mathbb{E}[\langle M \rangle_\infty] < \infty \Leftrightarrow M$ is a martingale s.t. $\sup_{t\geq 0} \mathbb{E}[M_t^2] < \infty$([3]*). In this case $(M_t^2 - \langle M \rangle_t,\ t \geq 0)$ is a UI martingale null in 0.*

Proof. □

Corollary. *Let M be a local continuous martingale, $M_0 = 0$ a.s. Then,*

$$\mathbb{P}\{\langle M \rangle_t = 0,\ \forall t \geq 0\} = 1 \quad \Leftrightarrow \quad \mathbb{P}\{M_t = 0,\ \forall t \geq 0\} = 1.$$

Proof. "$\Leftarrow$" is trivial. For the other implication, assume that $\langle M \rangle = 0$ a.s. then with (ii) of Theorem (4.9), M^2 is a UI martingale, and $\mathbb{E}[M_t^2] = \mathbb{E}[M_0^2] = 0$. □

Definition 4.2. Let M, N be local continuous martingale,

$$\begin{aligned}\langle M, N \rangle \quad &:= \quad \frac{1}{4}\left(\langle M + N \rangle_t - \langle M - N \rangle_t\right) \\ &= \quad \frac{1}{2}\left(\langle M + N \rangle_t - \langle M \rangle_t - \langle N \rangle_t\right).\end{aligned}$$

And in particular, $\langle M, M \rangle = \langle M \rangle$.

Proposition 4.10. *(i) $\langle M, N \rangle$ is the unique continuous adapted process at f.v.*[4]*, null in 0, s.t. $MN - \langle M, N \rangle$ is a local martingale.*

(ii) $(M, N) \mapsto \langle M, N \rangle$ is a symmetrical bi-linear application.

(iii) For all $t \geq 0$, $0 =: t_0^n < \cdots < t_{p_n}^n$, where the intervals go to 0,

$$\sum_{i=1}^{p_n} \left(M_{t_i^n} - M_{t_{i-1}^n}\right)\left(N_{t_i^n} - N_{t_{i-1}^n}\right) \quad \xrightarrow[n\to\infty]{\mathbb{P}} \quad \langle M, N \rangle_t.$$

(iv) Let T be a stopping time, $\langle M, N \rangle^T = \langle M^T, N \rangle = \langle M, N^T \rangle = \langle M^T, N^T \rangle$.

Proof. If we use the property of polarisation, *i.e.* $ab = \frac{1}{4}\left((a+b)^2 - (a-b)^2\right)$, we already proved everything. □

Remark. Let M, N be continuous martingales null in 0 s.t. $\mathbb{E}[\langle M \rangle_\infty] < \infty$, $\mathbb{E}[\langle N \rangle_\infty] < \infty$, then $MN - \langle M, N \rangle$ is a UI martingale.

Definition 4.3 (Orthogonality)**.** Two martingales are said to be orthogonal if $\langle M, N \rangle = 0$, *i.e.* MN is a local martingale.

Theorem 4.11 (Kunita–Watanabe inequality)**.** *Let M, N be two continuous local martingales, and H, K two measurable processes $(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$* ([5]*), then*

$$\begin{aligned}&\int_0^\infty |H_s||K_s||\mathrm{d}\langle M, N \rangle_s| \\ &\leq \sqrt{\int_0^\infty H_s^2\,\mathrm{d}\langle M \rangle_s} \cdot \sqrt{\int_0^\infty K_s^2\,\mathrm{d}\langle N \rangle_s}\end{aligned}$$

[2] In the sense that if (X_t) and (Y_t) follow this property then $\mathbb{P}\{X_t = Y_t,\ \forall t \geq 0\} = 1$.
[3] With the Doob inequality it is equivalent to have $\mathbb{E}[\sup_{t\geq 0} M_t^2] < \infty$.
[4] Finite variation.
[5] The process $(X_t,\ t \geq 0)$ is measurable if it is progressive.

Proof. In this proof we will use the notation $\langle M,N\rangle_s^t := \langle M,N\rangle_t - \langle M,N\rangle_s$, $s \le t$. Then with the Cauchy-Schwarz inequality we have (with theorem 4.7 and proposition 4.10 (iii)) for all $s < t$,

$$\left|\langle M,N\rangle_s^t\right| \quad \le \quad \sqrt{\langle M\rangle_s^t}\sqrt{\langle N\rangle_s^t}\,,$$

a.s. Then (4.1) hold a.s. for all $s < t \in \mathbb{Q}$, and with the continuity $\forall s < t \in \mathbb{R}$. We will now fix $\omega \in \Omega$ s.t. (4.1) is true.

Let us define $s =: t_0 < \cdots < t_p := t$ a subdivision,

$$\begin{aligned}\sum_{i=1}^p \left|\langle M,N\rangle_{t_{i-1}}^{t_i}\right| &\le \sum_{i=1}^p \sqrt{\langle M\rangle_{t_{i-1}}^{t_i}}\sqrt{\langle N\rangle_{t_{i-1}}^{t_i}} \\ &\le \sqrt{\sum_{i=1}^p \langle M\rangle_{t_{i-1}}^{t_i}}\sqrt{\sum_{i=1}^p \langle N\rangle_{t_{i-1}}^{t_i}} \\ &= \sqrt{\langle M\rangle_s^t\langle N\rangle_s^t}.\end{aligned}$$

By taking the supremum[6] on all the subdivisions of $[s,t]$,

$$\int_{[s,t]} |\mathrm{d}\langle M,N\rangle_u| \quad \le \quad \sqrt{\langle M\rangle_s^t\langle N\rangle_s^t}.$$

(...) ☐

4.4 Continuous semimartingale

Definition 4.4 (Semimartingale)**.** A process $(X_t,\ t \ge 0)$ is a continuous semimartingale if it can written as

$$X_t \quad = \quad X_0 \ + \ M_t \ + \ V_t\,,$$

where M is a local continuous martingale, V is a continuous adapted v.f. process, and $M_0 = V_0 = 0$ a.s.

Remark. We call this decomposition the (unique) canonical decomposition of X.

Definition 4.5. Let $X_t = X_0 + M_t + V_t$, $Y_t = Y_0 + N_t + W_t$ be two continuous semimartingales. We set

$$\langle X,Y\rangle_t \quad := \quad \langle M,N\rangle_t.$$

In particular $\langle X\rangle_t = \langle M\rangle_t$.

Proposition 4.12. *Let X, Y be two continuous semimartingales, and $0 =: t_0^n < \cdots < t_{p_n}^n$ a sequence of subdivision where the intervals go to 0 as $n \to \infty$, then*

$$\sum_{i=1}^{p_n}\left(X_{t_i^n} - X_{t_{i-1}^n}\right)\left(Y_{t_i^n} - Y_{t_{i-1}^n}\right) \quad \xrightarrow[n\to\infty]{\mathbb{P}} \quad \langle X,Y\rangle_t.$$

Proof. By polarisation we can prove the result with $X = Y$. Indeed,

$$\sum_{i=1}^{p_n}\left(X_{t_i^n} - X_{t_{i-1}^n}\right)^2 \quad = \quad I_n \ + \ J_n \ + \ K_n\,,$$

with,

$$\begin{cases} I_n &= \sum_{i=1}^{p_n}\left(M_{t_i^n} - M_{t_{i-1}^n}\right)^2 \\ J_n &= \sum_{i=1}^{p_n}\left(M_{t_i^n} - M_{t_{i-1}^n}\right)^2 \\ K_n &= 2\sum_{i=1}^{p_n}\left(M_{t_i^n} - M_{t_{i-1}^n}\right)\left(V_{t_i^n} - V_{t_{i-1}^n}\right).\end{cases}$$

we already have $I_n \xrightarrow[n\to\infty]{\mathbb{P}} \langle M\rangle_t = \langle X\rangle_t$. And

$$\begin{aligned}|J_n| \quad &\le \quad \underbrace{\max|V_{t_i^n} - V_{t_{i-1}^n}|}_{\xrightarrow{a.s.}0}\underbrace{\sum_{i=1}^{p_n}|V_{t_i^n} - V_{t_{i-1}^n}|}_{\le\int_0^t|\mathrm{d}V_s|<\infty} \\ &\xrightarrow{a.s.} \quad 0.\end{aligned}$$

Same kind of proof for K_n. ☐

5 Stochastic integral

5.1 Integration for bounded integral in L^2

We will first recall some concepts:

- Associativity: let $a : \mathbb{R}_+ \to \mathbb{R}$ continus and variation finite, $f\mathbb{R}_+ \to \mathbb{R}$ measurable, $\forall f$, $\int_0^t |f(s)||\mathrm{d}a(s)| < \infty$. We also have $b(t) := \int_0^t f(s)\,\mathrm{d}a(s)$ variation finite over $\mathbb{R}_+$. Then for all $g : \mathbb{R}_+ \to \mathbb{R}$ measure s.t. $\forall t \ge 0$, $\int_0^t |g(s)||f(s)||\mathrm{d}a(s)| < \infty$, $\forall t \ge 0$

$$\int_0^t g(s)\,\mathrm{d}b(s) \quad = \quad \int_0^t g(s)f(s)\,\mathrm{d}a(s).$$

- Integral stopping: $\forall T \ge 0,\ \forall t \ge 0$,

$$\begin{aligned}\int_0^{T\wedge t} f(s)\,\mathrm{d}a(s) \quad &= \quad \int_0^t f(s)\,\mathrm{d}a(s\wedge T) \\ &= \quad \int_0^t f(s)\mathbb{1}_{[0,T]}(s)\,\mathrm{d}a(s).\end{aligned}$$

- Change of variable: $A, \alpha : \mathbb{R}_+ \to \mathbb{R}$ continuous, growing, $A(0) = \alpha(0) = 0$. Then for all $f : \mathbb{R}_+ \to \mathbb{R}_+$ measure, $\forall t$,

$$\int_0^t f(\alpha(s))\,\mathrm{d}A(\alpha(s)) \quad = \quad \int_0^{\alpha(t)} f(u)\,\mathrm{d}A(u).$$

[6] Recall that

$$\sup_{t_i}\sum_{i=1}^p |a(t_i) - a(t_{i-1})| \quad = \quad \int_{[0,r]} |\mathrm{d}a|.$$

We know define the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. We use the notation

$$\mathbb{H}^2 := \left\{M : \text{cont. mart.}, \sup_{t\geq 0} \mathbb{E}[M_t^2] < \infty,\, M_0 = 0\right\}$$

Remark. It is equivalent to $\{M$ local martingale, $\mathbb{E}[\langle M\rangle_\infty] < \infty$, $M_0 = 0\}$ and $M^2 - \langle M\rangle$ martingale UI.

We observe that for T stopping time, $M \in \mathbb{H}^2 \Rightarrow M^T \in \mathbb{H}^2$. And for all $M, N \in \mathbb{H}^2$,

$$\begin{aligned} |\langle M, N\rangle_\infty| &\leq \int_0^\infty |\mathrm{d}\langle M, N\rangle_s| \\ &\leq^{(KW)} \sqrt{\int_0^\infty \mathrm{d}\langle M\rangle_s}\sqrt{\int_0^\infty \mathrm{d}\langle N\rangle_s} \\ &= \sqrt{\langle M\rangle_\infty}\sqrt{\langle N\rangle_\infty}, \end{aligned}$$

hence,

$$\mathbb{E}[|\langle M, N\rangle_\infty|] \leq^{(CS)} \sqrt{\mathbb{E}[\langle M\rangle_\infty]}\sqrt{\mathbb{E}[\langle N\rangle_\infty]} < \infty.$$

We will note $(M, N)_{\mathbb{H}^2} := \mathbb{E}[\langle M, N\rangle_\infty] \in \mathbb{R}$. We see that $(M, N)_{\mathbb{H}^2} = 0 \Rightarrow M = 0$. Actually $(M, N)_{\mathbb{H}^2}$ is a scalar product on $\mathbb{H}^2$. With the optimal stopping theorem we have $(M, N)_{\mathbb{H}^2} = \mathbb{E}[M_\infty N_\infty]$, and

$$\begin{aligned} \|M\|^2_{\mathbb{H}^2} &:= (M, M)_{\mathbb{H}^2} \\ &= \mathbb{E}[\langle M\rangle_\infty] = \mathbb{E}[M^2_\infty]. \end{aligned}$$

Proposition 5.1. *$(\mathbb{H}^2, (\cdot,\cdot)_{\mathbb{H}^2})$ is an Hilbert space.*

Proof. □

Now let us note, for all $M \in \mathbb{H}^2$,

$$\begin{aligned} L^2(M) &:= \left\{H \text{ prog. proc. } / \ \mathbb{E}\left[\int_0^\infty H_s^2 \,\mathrm{d}\langle M\rangle_s\right] < \infty\right\} \\ &= L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \mathrm{d}\mathbb{P}\,\mathrm{d}\langle M\rangle_s). \end{aligned}$$

Let $H, K \in L^2(M)$, we define

$$(H, K)_{L^2(M)} := \mathbb{E}\left[\int_0^\infty H_s K_s \,\mathrm{d}\langle M\rangle_s\right].$$

Theorem 5.2. *Let $M \in \mathbb{H}^2$, $\forall H \in L^2(M)$, $\exists! H \cdot M \in \mathbb{H}^2$ s.t.*

$$\langle H\cdot M, N\rangle = \underbrace{H\cdot\langle M, N\rangle}_{:= \left(\int_0^t H_s \,\mathrm{d}\langle M,N\rangle_s,\, t\geq 0\right)},$$

for all $N \in \mathbb{H}^2$.

The application

$$\begin{aligned} L^2(M) &\to \mathbb{H}^2 \\ H &\mapsto H\cdot M \end{aligned}$$

is an isometry (linear that preserves the norm).

We call $H \cdot M = \left((H\cdot M)_t := \int_0^t H_s \,\mathrm{d}M_s,\, t \geq 0\right)$ the stochastic integral (or Itô's integral).

Proof. □

Proposition 5.3 (Associativity)**.** *Let $M \in \mathbb{H}^2$, $K \in L^2(M)$, we can then define $H \in L^2(K\cdot M)$. Then $HK \in L^2(M)$, we can define $HK \cdot M \in L^2(M)$, and*

$$HK\cdot M = H\cdot(K\cdot M).$$

Proof. □

Remark. Then we can write

(i)
$$\int_0^t H_s(K_s \,\mathrm{d}M_s) = \int_0^t H_s K_s \,\mathrm{d}M_s.$$

(ii) $\forall M, N \in \mathbb{H}^2$, $\forall H \in L^2(M)$, $\forall K \in L^2(N)$,

$$\begin{aligned} \left\langle \int_0^\cdot H_s \,\mathrm{d}M_s, N\right\rangle_t &= \int_0^t H_s \,\mathrm{d}\langle M, N\rangle_s\ ; \\ \left\langle \int_0^\cdot H_s \,\mathrm{d}M_s, \int_0^\cdot K_s \,\mathrm{d}N_s\right\rangle_t &= \int_0^t H_s K_s \,\mathrm{d}\langle M, N\rangle_s\ ; \\ \left\langle \int_0^\cdot H_s \,\mathrm{d}M_s\right\rangle_t &= \int_0^t H_s^2 \,\mathrm{d}\langle M\rangle_s. \end{aligned}$$

Proposition 5.4. *Let $M \in \mathbb{H}^2$, $H \in L^2(M)$ and T stopping time, then*

$$\begin{aligned} (H\cdot M)^T &= H\cdot M^T \\ &= H\mathbb{1}_{[0,T]}\cdot M. \end{aligned}$$

Proof. □

5.2 Integration for continuous semimartingales

Let M be a continuous local martingale, we define

$$L^2_{loc}(M) := \left\{H \text{ prog. proc. } / \ \mathbb{E}\left[\int_0^\infty H_s^2 \,\mathrm{d}\langle M\rangle_s\right] < \infty\right\}.$$

Theorem 5.5. *Let M be a continuous local martingale, $H \in L^2_{loc}(M)$. Then*

(i) $\exists! H\cdot M$ continuous local martingale, null in zero s.t.

$$\langle H\cdot M, N\rangle = H\cdot\langle M, N\rangle,$$

for all N local martingale.

(ii) With T stopping time, $H\cdot M^T = (H\cdot M)^T = H\mathbb{1}_{[0,T]}\cdot M$.

(iii) If $M \in \mathbb{H}^2$, $H \in L^2(M)$, then $H\cdot M$ is the Itô's integral defined in the last section.

Proof. □

Remark. With M continuous local martingale, $H \in L^2_{loc}(M)$ and T stopping time,

(i) if $\mathbb{E}\left[\int_0^T H_s^2 \,\mathrm{d}\langle M\rangle_s\right] < \infty$, then

$$\mathbb{E}\left[\int_0^T H_s \,\mathrm{d}M_s\right] = 0 \;;$$
$$\mathbb{E}\left[\left(\int_0^T H_s \,\mathrm{d}M_s\right)^2\right] = \mathbb{E}\left[\int_0^T H_s^2 \,\mathrm{d}\langle M\rangle_s\right] \;;$$

(ii) if $\forall t \geq 0$, $\mathbb{E}\left[\int_0^T H_s^2 \,\mathrm{d}\langle M\rangle_s\right] < \infty$, then $\left(\int_0^t H_s \,\mathrm{d}M_s\, t \geq 0\right)$ is a continuous martingale square integrable, null in zero, s.t. $\forall t \geq 0$,

$$\mathbb{E}\left[\int_0^t H_s \,\mathrm{d}M_s\right] = 0 \;;$$
$$\mathbb{E}\left[\left(\int_0^t H_s \,\mathrm{d}M_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 \,\mathrm{d}\langle M\rangle_s\right].$$

We say that H is a process that is locally bounded if for all t,

$$\sup_{s\in[0,t]} |H_s| < \infty \quad \text{a.s.}$$

Definition 5.1. Let $X = X_0 + M + V$ be a continuous semimartingale, and H a progressive process locally bounded, then we define

$$H \cdot X \;:=\; H \cdot M \;+\; H \cdot V.$$

Remark. Here $H \cdot M$ is a local continuous martingale, null in 0. And $H \cdot V$ is an adapted continuous process, with finite variations, null in 0.

Proposition 5.6. *Here are few properties that we already saw. Let H, K progressive processes locally bounded, and X continuous semimartingale.*

(i) $H \cdot (K \cdot X) = HK \cdot X$.

(ii) Let T be a stopping time, then $(H \cdot X)^T = H \cdot X^T = H\mathbb{1}_{[0,T]} \cdot X$.

(iii) If X is a continuous local martingale (or process with finite variations), then it is the same for $H \cdot X$.

(iv) $(H, X) \mapsto H \cdot X$ *is bilinear.*

(v) Let H progressive s.t. $H_s(\omega) = \sum_{i=0}^{p-1} H^{(i)}(\omega)\mathbb{1}_{]t_i,t_{i+1}]}(s)$, *where* $0 =: t_0 < \cdots < t_p$, $\forall i$ $H^{(i)}$ *is* $(\mathcal{F}_{t_i})$*-measurable, then*

$$(H \cdot X)_t = \sum_{i=0}^{p-1} H^{(i)}(X_{t_{i+1}\wedge t} - X_{t_i\wedge t}).$$

Proposition 5.7. *Let X a continuous semimartingale and H a continuous adapted process, then $\forall t > 0$, $\forall 0 =: t_0^n < \ldots t_{p_n}^n := t$ where the interval goes to zero with $n \to \infty$,*

$$\int_0^t H_s \,\mathrm{d}X_s \;\overset{\mathbb{P}}{=}\; \lim_{n\to\infty} \sum_{i=0}^{p_n-1} H_{t_i^n}\left(X_{t_{i+1}^n} - X_{t_i^n}\right).$$

Proof. □

Remark. We have to be careful, it's wrong to replace $H_{t_i^n}$ by $H_{t_{i+1}^n}$ or any other value in $]H_{t_i^n}, H_{t_{i+1}^n}]$ in the equality.

Proposition 5.8 (Integration by parts)**.** *Let X, Y continuous semimartingales, then*

$$X_tY_t = X_0Y_0 + \int_0^t X_s \,\mathrm{d}Y_s + \int_0^t Y_s \,\mathrm{d}X_s + \langle X, Y\rangle_t$$

for all $t \geq 0$.

Proof. □

Remark. With M local continuous martingale,

$$M_t^2 = M_0^2 + 2\int_0^t M_s \,\mathrm{d}M_s + \langle M\rangle_t.$$

6 Itô's formula and applications

6.1 Itô's formula

Theorem 6.1 (Itô's formula)**.** *(i) (Unidimensional) Let X be a continuous semimartingale, $f : \mathbb{R} \to \mathbb{R}$ in $\mathcal{C}^2$, then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \,\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s) \,\mathrm{d}\langle X\rangle_s.$$

(ii) (Multidimensional) Let $X^1, \ldots, X^N$ be continuous semimartingales, $F : \mathbb{R}^N \to \mathbb{R}$ in $\mathcal{C}^2$,

$$F(X_t) = F(X_0) + \sum_{i=1}^N \int_0^t \frac{\partial F}{\partial x_i}(X_s) \,\mathrm{d}X_s^i + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s) \,\mathrm{d}\langle X^i, X^j\rangle_s.$$

Proof. □

Remark. (i) In the case $F(x, y) := xy$ we found the formula of the integration by parts.

(ii) The Itô's formula is still true if $(X_t,\, t \geq 0)$ with value in $D \subset \mathbb{R}^N$ an open set (and convex), and if $F : D \to \mathbb{R}$ in $\mathcal{C}^2$.

(iii) If $X^1, \ldots, X^k$ are continuous, adapted and with finite variations, the formula is still true if

$$F \in \mathcal{C}^{\overbrace{1,\ldots,1}^{k},\overbrace{2,\ldots,2}^{N-k}}$$

(iv) The differential version is

$$\mathrm{d}f(X_t) \;=\; f'(X_t)\,\mathrm{d}X_t \;+\; \frac{1}{2}f''(X_t)\,\mathrm{d}\langle X\rangle_t$$

Example 6.1 (Multidimensional brownian motion)**.** We will note $n \geq 1$ the dimension, $\left(B := \left(B_t^{(1)}, \dots, B_t^{(n)}\right),\, t \geq 0\right)$ a $(\mathcal{F}_t)$-brownian motion in $\mathbb{R}^n$ (hence $B^{(1)}, \dots, B^{(n)}$ are independant $\mathcal{F}_t$-brownian motions).

Let us first see the case $n = 1$, $f : \mathbb{R} \to \mathbb{R}$ in $\mathcal{C}^2$,

$$f(B_t) \;=\; f(0) \;+\; \int_0^t f'(B_s)\,\mathrm{d}B_s \;+\; \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s.$$

Now we take $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \in \mathcal{C}^{1,2}$, hence

$$F(t, B_t) \;=\; F(0,0) \;+\; \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right)(s, B_s)\,\mathrm{d}s \;+\; \int_0^t \frac{\partial F}{\partial x}(s, B_s)\,\mathrm{d}B_s.$$

So if $\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0$ we have $(F(t, B_t),\, t \geq 0)$ is a local martingale[7]. It is the case for $F_1(t,x) = x$, $F_2(t,x) = x^2 - t$, $F_3(t,x) = x^3 - 3tx$, *etc.* More generally

$$H_n(x) \;:=\; (-1)^n e^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-\frac{x^2}{2}}\right) \;;$$

$$H_n(x,t) \;:=\; t^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{t}}\right) \qquad \text{(mod. Hermite's poly.)}$$

and then $\forall n$, $(H_n(B_t, t),\, t \geq 0)$ is a continuous local martingale. We also have, for all $t \geq 0$,

$$\mathbb{E}\left[\int_0^t \left|\frac{\partial H_n}{\partial x}(B_s, s)\right|^2 \mathrm{d}s\right] \;<\; \infty$$

sp we know that $\forall n$, $(H_n(B_t, t),\, t \geq 0)$ is a martingale[8].

More generally, if B is a $(\mathcal{F}_t)$-brownian motion in $\mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}$ in $\mathcal{C}^2$,

$$F(B_t) \;=\; F(0) \;+\; \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(B_s)\,\mathrm{d}B_s^i \;+\; \frac{1}{2}\int_0^t \Delta F(B_s)\,\mathrm{d}s.$$

6.2 Exponential semimartingale

Theorem 6.2. *Let X be a continuous semimartingale, then $\exists! Z$ continuous semimartingale s.t.*

$$Z_t \;=\; e^{X_0} \;+\; \int_0^t Z_s\,\mathrm{d}X_s\,, \qquad t \geq 0.$$

Moreover,

$$Z_T \;=\; \mathcal{E}(X)_t \;=\; e^{X_t - \frac{1}{2}\langle X\rangle_t}.$$

Proof. □

Proposition 6.3. *Let M be a continuous local martingale, $\lambda \in \mathbb{C}$, then*

$$\mathcal{E}(\lambda M)_t \;:=\; \exp\left(\lambda M_t - \frac{\lambda^2}{2}\langle M\rangle_t\right),$$

$t \geq 0$, is a continuous local martingale $\mathbb{C}$-valued.

Proof. □

Let L be a continuous local martingale, $L_0 = 0$. Then $\mathcal{E}(L)_t$ is a positive continuous local martingale, $\mathcal{E}(L)_0 = 1$. Hence $\mathcal{E}(L)$ is a positive supermartingale and with Fatou's Lemma $\mathbb{E}[\mathcal{E}(L)_\infty] \leq 1$. Now we want to know if $\mathbb{E}[\mathcal{E}(L)_\infty] = 1$, *i.e.*[9] $\mathcal{E}(L)$ is a martingale UI.

Theorem 6.4. *Let L be a continuous local martingale, $L_0 = 0$ a.s., then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).*

(i) (Novikov) $\mathbb{E}\left[e^{\frac{1}{2}\langle L\rangle_\infty}\right] < \infty$.

(ii) (Kazamaki) L continuous martingale UI, $\mathbb{E}\left[e^{\frac{1}{2}L_\infty}\right] < \infty$.

(iii) $\mathbb{E}[\mathcal{E}(L)_\infty] = 1$.

Proof. □

6.3 Levy's characterization of the Brownian motion

We are in a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Let $B = (B^1, \dots, B^n)$ a $\mathcal{F}_t$-brownian motion $\mathbb{R}^n$-valued, then

$$\langle B^i, B^j\rangle_t \;=\; t\mathbb{1}_{\{i=j\}}.$$

Theorem 6.5 (Levy)**.** *(i) M continuous local martingale, $M_0 = 0$ a.s., then*

$$\langle M\rangle_t \;=\; t\,,\ \forall t \geq 0 \;\;\Rightarrow\;\; M \text{ is a } (\mathcal{F}_t)\text{-brownian motion.}$$

(ii) $M^1, \dots, M^n$ continuous local martingales null in 0, then

$$\langle M^i, M^j\rangle_t \;=\; t\mathbb{1}_{\{i=j\}} \;\;\Rightarrow\;\; M \text{ is a } (\mathcal{F}_t)\text{-brownian motion } \mathbb{R}^n\text{-valued.}$$

Proof. □

[7] Indeed we know that $\int \cdot\,\mathrm{d}B_s$ is a local martingale.

[8] Indeed recall that if $\int_0^t H_s\,\mathrm{d}M_s$ and if $\forall t > 0$, $\mathbb{E}\left[\int_0^t H_s^2\,\mathrm{d}\langle M\rangle_s\right] < \infty$, then $\left(\int_0^t H_s\,\mathrm{d}M_s,\, t \geq 0\right)$ is a martingale square integrable.

[9] Indeed recall that if M is a continuous positive supermartingale and $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$, then M is a martingale UI.

Example 6.2. Let B a $(\mathcal{F}_t)$-brownian motion, and

$$\beta_t \quad := \quad \int_0^t \operatorname{sgn}(B_s)\,\mathrm{d}B_s \qquad t \geq 0\,,$$

then β is a continuous local martingale, $\beta_0 = 0$, $\langle\beta\rangle_t = \int_0^t \operatorname{sgn}^2(B_s)\,\mathrm{d}s = t$. And with Levy's Theorem β is a $(\mathcal{F}_t)$-brownian motion.

Example 6.3. Let (X,Y) two brownian motions $\mathbb{R}^2$-valued, $X_0 = Y_0 = 0$. For $\theta \in \mathbb{R}$,

$$\begin{aligned} X_t^\theta \quad &:= \quad X_t \cos\theta \;-\; Y_t \sin\theta\;; \\ Y_t^\theta \quad &:= \quad X_t \sin\theta \;+\; Y_t \cos\theta\,, \qquad t \geq 0. \end{aligned}$$

Hence X^θ, Y^θ are two continuous martingales null in 0 and $\langle X^\theta\rangle_t = \langle Y^\theta\rangle_t = t$, $\langle X^\theta, Y^\theta\rangle_t = 0$, then (X^θ, Y^θ) is a brownian motion.

More generally let B a $(\mathcal{F}_t)$-brownian motion, $A \in \mathcal{O}_n$, then $(AB_t,\, t \geq 0)$ is a brownian motion.

6.4 Dambis–Dubing–Schwarz Theorem

Theorem 6.6 (Dambis–Dubing–Schwarz)**.** *Let M be a continuous local martingale null in 0, then*

$$M_t \quad = \quad B_{\langle M\rangle_t}\,, \qquad t \geq 0$$

with B a brownian motion.

Proof. □

Remark. B is not a $(\mathcal{F}_t)$-brownian motion but a $(\mathcal{F}_{\tau_r})$-brownian motion.

Theorem 6.7 (Knight)**.** *Let $M^1, \ldots, M^n$ continuous local martingales, null in 0, $\langle M^i, M^j\rangle = 0$ for $i \neq j$, then*

$$\forall 1 \leq i \leq n\,, \quad M_t^i \quad = \quad B^i_{\langle M^i\rangle_t}\,, \qquad t \geq 0\,,$$

$(B^1, \ldots, B^n)$ is a brownian motion $\mathbb{R}^n$-valued.

6.5 Examples: multidimensional brownian motion

Example 6.4. Let M be a continuous local martingale, $M_0 = 0$ a.s., then

(i) $\mathbb{P}\{\lim_{t\to\infty} |M_t| = \infty\} \;=\; 0.$

(ii) $\{\lim_{t\to\infty} M_t$ exists (is finite)$\} \;=\; \{\langle M\rangle_\infty < \infty\} = \{\sup_{t\geq 0} M_t < \infty$ or $\inf_{t\geq 0} M_t > -\infty\}$ a.s.

(iii) $\{\langle M\rangle_\infty = \infty\} = \{\limsup_{t\to\infty} M_t = \infty,\, \liminf_{t\to\infty} M_t = -\infty\}$ a.s.

[Proof to be written]

Example 6.5 (Polar points and 2-d brownian motion)**.** Let (β,γ) be a brownian motion $\mathbb{R}^2$-valued, $\beta_0 = \gamma_0 = 0$. We define $M_t = e^{\beta_t}\cos\gamma_t$, $N_t = e^{\beta_t}\sin\gamma_t$. With Itô we have

$$\begin{aligned} \mathrm{d}M_t \quad &= \quad M_t\,\mathrm{d}\beta_t \;-\; N_t\,\mathrm{d}\gamma_t\;; \\ \mathrm{d}N_t \quad &= \quad N_t\,\mathrm{d}\beta_t \;+\; M_t\,\mathrm{d}\gamma_t\,, \end{aligned}$$

so M, N are local martingales, and

$$\begin{aligned} \mathrm{d}\langle M\rangle_t \quad &= \quad e^{2\beta_t}\,\mathrm{d}t\;; \\ \mathrm{d}\langle N\rangle_t \quad &= \quad e^{2\beta_t}\,\mathrm{d}t\;; \\ \mathrm{d}\langle M,N\rangle_t \quad &= \quad 0. \end{aligned}$$

Then with the Knight's Theorem[10], $(M_t - 1, N_t) = B_{\int_0^t e^{2\beta_t}\,\mathrm{d}s}$ with B a brownian motion $\mathbb{R}^2$-valued. Let us take $\omega \in \{\langle M\rangle_\infty\}$, then $\lim_{t\to\infty} M_t$ and $\lim_{t\to\infty} N_t$ exist and are finite, hence $\lim_{t\to\infty}(M_t^2 + N_t^2) = \lim_{t\to\infty} e^{2\beta_t}$ exists and is finite. So we conclude that $\langle M\rangle_\infty = \infty$ a.s., $\int_0^\infty e^{2\beta_s}\,\mathrm{d}s = \infty$ a.s. So we can write $(M_t, N_t) = B_{\langle M\rangle_t} + (1,0)$, and as $|(M_t,N_t)| = e^{\beta_t} > 0$,

$$\begin{aligned} &\mathbb{P}\{\exists t \geq 0\,,\; B_{\langle M\rangle_t} = (-1,0)\} \;=\; 0 \\ \Rightarrow\quad &\mathbb{P}\{\exists s \geq 0\,,\; B_s = (-1,0)\} \;=\; 0 \\ \Rightarrow\quad &\forall a \in \mathbb{R}^2 \setminus \{0\}\,,\; \mathbb{P}\{\exists s \geq 0\,,\; B_s = 0\} \;=\; 0 \end{aligned}$$

with rotation and scaling.

Example 6.6 (3-d brownian motion)**.** Let B be a brownian motion $\mathbb{R}^3$-valued, then $\lim_{t\to\infty} |B_t| = \infty$ a.s.

To show that we just have to show that $\forall x \in \mathbb{R}^3 \setminus \{0\}$, $\lim_{t\to\infty} |B_t + x| = \infty$ a.s. We define $Z_t := |B_t + x|^2 = \sum_{i=1}^3 \left(B_t^i + x_i\right)^2$, and with Itô's formula,

$$\begin{aligned} \mathrm{d}Z_t \quad &= \quad \sum_{i=1}^3 2(B_t^i + x_i)\,\mathrm{d}B_t^i \;+\; \frac{1}{2}\sum_{i=1}^3 2\,\mathrm{d}t \\ &= \quad 2\sum_{i=1}^3 (B_t^i + x_i)\,\mathrm{d}B_t^i \;+\; 3\,\mathrm{d}t. \end{aligned}$$

We define $f : \mathbb{R}_+^* \to \mathbb{R}$, $x \mapsto \frac{1}{\sqrt{x}}$, $f \in \mathcal{C}^2$, and $Y_t := f(Z_t)$, with Itô,

$$\begin{aligned} Y_t \quad &= \quad Y_0 \;-\; \frac{1}{2}\int_0^t \frac{1}{Z_t^{\frac{3}{2}}}\,\mathrm{d}Z_s \;+\; \frac{1}{2}\frac{3}{4}\int_0^t \frac{1}{Z_s^{\frac{5}{2}}}\,\mathrm{d}\langle Z\rangle_s \\ &= \quad [\text{loc. mart.}] \;-\; \frac{3}{2}\int_0^t \frac{1}{Z_s^{\frac{3}{2}}}\,\mathrm{d}s \;+\; \frac{3}{8}\int_0^t 2^2\,\frac{Z_s}{Z_s^{\frac{5}{2}}}\,\mathrm{d}s \\ &= \quad [\text{loc. mart.}]. \end{aligned}$$

So Y is a positive local martingale and $\mathbb{E}[Y_0] = \frac{1}{|x|} < \infty$, hence Y is a positive supermartingale, $\lim_{t\to\infty} Y_t = \xi \geq 0$ a.s. So $\lim_{t\to\infty} |B_t + x| = \frac{1}{\xi}$ a.s., and we know that $\limsup_{t\to\infty} |B_t + x| = \infty$ a.s., then $\lim_{t\to\infty} |B_t + x| = \infty$ a.s.

[10] We have to be careful, $M_0 = 1 \neq 0$.

Example 6.7. Let B be a brownian motion $\mathbb{R}^n$-valued, $n \geq 2$, with $B_0 = x \in \mathbb{R}^n \setminus \{0\}$. We set $Z_t := |B_t|^2$, hence,

$$\mathrm{d}Z_t \;=\; 2\sum_{i=1}^n B_t^i \,\mathrm{d}B_t^i \;+\; n\,\mathrm{d}t.$$

Now with $f : \mathbb{R}_+^* \to \mathbb{R} \in \mathcal{C}^2$, $Y_t := f(Z_t)$,

$$\begin{aligned}\mathrm{d}Y_t &= f'(Z_t)\,\mathrm{d}Z_t \;+\; \frac{1}{2}f''(Z_t)\,\mathrm{d}\langle Z\rangle_t \\ &= [\text{loc. mart.}] \;+\; nf'(Z_t)\,\mathrm{d}t \;+\; \frac{1}{2}f''(Z_t)\times 4Z_t\,\mathrm{d}t.\end{aligned}$$

So if $f'(y) + \frac{2}{n}yf''(y) = 0$, $\forall y \in \mathbb{R}_+^*$, then Y is a local martingale. The following functions have this last property:

$$\begin{aligned} f(y) &:= \frac{1}{2}\ln y\,, \qquad \text{for } n=2,\ y>0\ ; \\ f(y) &:= y^{1-\frac{n}{2}}\,, \qquad \text{for } n\geq 3,\ y>0. \end{aligned}$$

Now we use the notation $T_a := \inf\{t \geq 0 : |B_t| = a\}$, $a > 0$. And define $0 < r < |x| < R$, then $(Y_{t\wedge T_R\wedge T_r},\, t \geq 0)$ is a continuous bounded local martingale, so a martingale UI. With the optimal stopping theorem we have $\mathbb{E}\left[f(Z_{T_R\wedge T_r})\right] = f(|x|^2)$. With $n = 2$,

$$\begin{aligned} & \mathbb{E}\left[\ln|B_{T_R\wedge T_r}|\right] \;=\; \ln|x| \\ \Leftrightarrow\quad & \mathbb{E}\left[(\ln R)\mathbb{1}_{\{T_R<T_r\}}\right] \;+\; \mathbb{E}\left[(\ln r)\mathbb{1}_{\{T_r<T_R\}}\right] \;=\; \ln|x| \\ \Leftrightarrow\quad & (\ln R)\mathbb{P}\{T_R<T_r\} \;+\; (\ln r)\mathbb{P}\{T_r<T_R\} \;=\; \ln|x| \\ \Leftrightarrow\quad & \mathbb{P}\{T_r<T_R\} \;=\; \frac{\ln R - \ln|x|}{\ln R - \ln r}. \end{aligned}$$

So with $R \to \infty$, $\mathbb{P}\{T_r < \infty\} = 1$.

Now with $n \geq 3$

$$\begin{aligned} & \mathbb{E}\left[|B_{T_R\wedge T_r}|^{2-n}\right] \;=\; |x|^{2-n} \\ \Leftrightarrow\quad & R^{2-n}\mathbb{P}\{T_R<T_r\} \;+\; r^{2-n}\mathbb{P}\{T_r<T_R\} \;=\; |x|^{2-n} \\ \Leftrightarrow\quad & \mathbb{P}\{T_r - T_R\} \;=\; \frac{|x|^{2-n} - R^{2-n}}{r^{2-n} - R^{2-n}}\,, \end{aligned}$$

with $R \to \infty$, $\mathbb{P}\{T_r < \infty\} = \left(\frac{r}{|x|}\right)^{n-2} < 1$.

6.6 Burkholder–Davis–Gundy inequality

We are in the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. And for X process, we use the notation

$$X_t^* \;:=\; \sup_{s\in[0,t]}|X_s|\,, \qquad \forall t \in \mathbb{R}_+ \cup \{\infty\}.$$

Theorem 6.8 (Burkholder–Davis–Gundy)**.** *Let $p \in \mathbb{R}_+^*$, then $\exists 0 < c_p \leq C_p < \infty$ s.t. $\forall M$ continuous local martingale, $M_0 = 0$ a.s.,*

$$c_p\mathbb{E}\left[\langle M\rangle_\infty^{\frac{p}{2}}\right] \;\leq\; \mathbb{E}\left[(M_\infty^*)^p\right] \;\leq\; C_p\mathbb{E}\left[\langle M\rangle_\infty^{\frac{p}{2}}\right].$$

And in particular, $\forall T$ stopping time,

$$c_p\mathbb{E}\left[\langle M\rangle_T^{\frac{p}{2}}\right] \;\leq\; \mathbb{E}\left[(M_T^*)^p\right] \;\leq\; C_p\mathbb{E}\left[\langle M\rangle_T^{\frac{p}{2}}\right].$$

Lemma 6.9. *Let B be a $(\mathcal{F}_t)$-brownian motion, T a stopping time, $\beta > 1$, $\delta > 0$, $x > 0$, then*

$$\begin{aligned} \mathbb{P}\left\{B_T^* > \beta x,\, \sqrt{T} \leq \delta x\right\} &\leq \frac{\delta^2}{(\beta-1)^2}\mathbb{P}\left\{B_T^* \geq x\right\}\ ; \\ \mathbb{P}\left\{\sqrt{T} > \beta x,\, B_T^* \leq \delta x\right\} &\leq \frac{\delta^2}{\beta^2-1}\mathbb{P}\left\{\sqrt{T} \geq x\right\}. \end{aligned}$$

Lemma 6.10. *Let $\xi \geq 0$, $\eta \geq 0$, r.v. s.t.*

$$\mathbb{P}\{\xi > 2x,\, \eta \leq \delta x\} \;\leq\; \delta^2\mathbb{P}\{\xi \geq x\}\,, \qquad \forall \delta, x > 0\,,$$

then, $\forall p > 0$, $\exists c(p) < \infty$ s.t.

$$\mathbb{E}\left[\xi^p\right] \;\leq\; c(p)\mathbb{E}[\eta^p].$$

Proof. (Theorem 6.8) ☐

Proof. (Lemma 6.9) ☐

Proof. (Lemma 6.10) ☐

Example 6.8. Let M be a continuous local martingale, $M_0 = 0$ a.s.,

$$\begin{aligned} \mathbb{E}[\langle M\rangle_\infty] < \infty \quad &\Leftrightarrow\quad \mathbb{E}\left[(M_\infty^*)^2\right] < \infty \qquad \text{(BDG)} \\ &\Leftrightarrow\quad M \text{ martingale bounded in } L^2. \end{aligned}$$

Example 6.9 (Wald identities)**.** Let B be a $(\mathcal{F}_t)$-brownian motion, T a stopping time,

$$\begin{aligned} \mathbb{E}\left[\sqrt{T}\right] < \infty \quad &\Leftrightarrow\quad \mathbb{E}[\underbrace{B_T^*}_{=(B^T)_\infty^*}] < \infty \qquad \text{(BDG)} \\ &\Rightarrow\quad B^T \text{ martingale UI} \\ &\Rightarrow\quad \mathbb{E}[B_T] = \mathbb{E}[B_0] = 0 \qquad \text{(Opt. stopping)} \end{aligned}$$

$$\begin{aligned} \mathbb{E}[T] < \infty \quad &\Leftrightarrow\quad \mathbb{E}\left[(B_T^*)^2\right] < \infty \qquad \text{(BDG)} \\ &\Rightarrow\quad B^T,\ \left((B_t^T)^2 - (T\wedge t),\, t \geq 0\right) \text{ mart. UI} \\ &\Rightarrow\quad \mathbb{E}\left[B_T^2 - T\right] = 0 \\ &\Rightarrow\quad \mathbb{E}\left[B_T^2\right] = \mathbb{E}[T]. \end{aligned}$$

6.7 Martingales of a brownian filtration

We are in the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let B be a brownian motion. And we set $(\mathcal{F}_t)$ (the usually augmentation of) the canonical filtration of B.

Theorem 6.11. *Let $(\mathcal{F}_t)$ be (the usually augmentation of) the canonical filtration of B, then for all M continuous local martingale, there exists a unique constant c and a progressive process $(H_t,\, t \geq 0)$ s.t. $\forall t > 0$, $\int_0^t H_s^2\,\mathrm{d}s < \infty$ a.s., s.t.*

$$M_t \;=\; c \;+\; \int_0^t H_s\,\mathrm{d}B_s.$$

If moreover M is a continuous martingale with $\sup_{t\geq 0}\mathbb{E}[M_t^2] < \infty$ then

$$\mathbb{E}\left[\int_0^\infty H_s^2\,\mathrm{d}s\right] \;<\; \infty.$$

6.8 Girsanov theorem

We are in the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

Theorem 6.12 (Girsanov)**.** *Define $(L_t,\, t \geq 0)$ a continuous local martingale, $L_0 = 0$ a.s. Assume that $\mathbb{E}[\mathcal{E}(L)_\infty] = 1$ (i.e. $\mathcal{E}(L)$ is a martingale UI). Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F}_\infty)$ defined by $\mathbb{Q} := \mathcal{E}(L)_\infty \cdot \mathbb{P}$ (i.e. $\forall A \in \mathcal{F}_\infty$, $\mathbb{Q} = \mathbb{E}[\mathcal{E}(L)_\infty \mathbb{1}_A]$). Then for all M local continuous $\mathbb{P}$-martingale,*

$$M \;-\; \langle M, L\rangle$$

is a local continuous $\mathbb{Q}$-martingale.

Proof. □

Remark. (i) $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F}_\infty$, but not the inverse. And we have $\forall t \geq 0$, $\mathcal{Q} \sim \mathcal{P}$ on $\mathcal{F}_t$.

(ii) If X is a $\mathbb{P}$-semimartingale, it's also a $\mathbb{Q}$-semimartingale.

(iii) A result $\mathbb{P}$-a.s. or in probability hold un $\mathbb{Q}$.

(iv) Set B a $(\mathcal{F}_t)$-brownian motion under $\mathbb{P}$, then $B - \langle B, L\rangle$ is a $(\mathcal{F}_t)$-brownian motion under $\mathbb{Q}$.

Theorem 6.13 (Girsanov, horizon finite version)**.** *For $t > 0$, let $(L_s,\, s \in [0,t])$ be a continuous local martingale, $L_0 = 0$ a.s. Assume that $\mathbb{E}\left[e^{L_t - \frac{1}{2}\langle L\rangle_t}\right] = 1$. Let $\mathbb{Q}$ on $(\Omega, \mathcal{F}_t)$ be the probability measure defined by $\mathbb{Q} = e^{L_t - \frac{1}{2}\langle L\rangle_t} \cdot \mathbb{P}$. Then for all M local $\mathbb{P}$-martingale, the process $(M_s - \langle M, L\rangle_s,\, s \in [0,t])$ is a local $\mathbb{Q}$-martingale.*

Example 6.10 (Cameron–Martin)**.** Let $h : \mathbb{R}_+ \to \mathbb{R}$ measurable s.t. $\forall t$, $\int_0^t h^2(s)\,\mathrm{d}s < \infty$. We define $L_t := \int_0^t h(s)\,\mathrm{d}B_s$, $t \geq 0$. Then $(\mathcal{E}(L)_t,\, t \geq 0)$ is a positive supermartingale and as $\mathbb{E}[e^{\frac{1}{2}\langle L\rangle_t}] < \infty$, $\forall t > 0$, $(\mathcal{E}(L)_s,\, s \in [0,t])$ is a martingale UI.

There exists[11] a probability $\mathbb{Q}$ s.t. $\forall t$, $\mathbb{Q}_{|\mathcal{F}_t} = \mathcal{E}(L)_t \cdot \mathbb{P}_{|\mathcal{F}_t}$. With the Girsanov theorem $\forall t \geq 0$, $(B_s - \int_0^t h(u)\,\mathrm{d}u,\, s \in [0,t])$ is a $\mathbb{Q}$-brownian motion. Then $(B_s - \int_0^t h(u)\,\mathrm{d}u,\, s \geq 0)$ is a $\mathbb{Q}$-brownian motion.

In particular let $h(t) = \gamma \in \mathbb{R}$, then $\mathbb{Q}_{|\mathcal{F}_t} = e^{\gamma B_t - \frac{1}{2}\gamma^2 t} \cdot \mathbb{P}_{|\mathcal{F}_t}$. The process $(B_t - \gamma t,\, t \geq 0)$ is a $\mathbb{Q}$-brownian motion and then B under $\mathbb{Q}$ is a brownian motion with drift γ.

We are looking at $T_a := \inf\{t \geq 0 : B_t = a\}$, for all $t \geq 0$,

$$\begin{aligned}
\mathbb{Q}\{T_a \leq t\} &= \mathbb{E}\left[e^{\gamma B_t - \frac{1}{2}\gamma^2 t}\mathbb{1}_{\{T_a \leq t\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\gamma B_t - \frac{1}{2}\gamma^2 t}\mathbb{1}_{\{T_a \leq t\}}\,\middle|\,\mathcal{F}_{T_a \wedge t}\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{T_a \leq t\}}\mathbb{E}\left[e^{\gamma B_t - \frac{1}{2}\gamma^2 t}\,\middle|\,\mathcal{F}_{T_a \wedge t}\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{T_a \leq t\}} e^{\gamma B_{T_a \wedge t} - \frac{1}{2}\gamma^2 (T_a \wedge t)}\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\{T_a \leq t\}} e^{\gamma a - \frac{1}{2}\gamma^2 (T_a \wedge t)}\right], \qquad T_a \sim \frac{a^2}{B_1^2} \\
&= \int_0^t e^{\gamma a - \frac{1}{2}\gamma^2 s}\frac{|a|}{\sqrt{2\pi s^3}} e^{-\frac{a^2}{2s}}\,\mathrm{d}s \\
&= \int_0^t \frac{|a|}{\sqrt{2\pi s^3}} e^{-\frac{(\gamma s - a)^2}{2s}}\,\mathrm{d}s.
\end{aligned}$$

Now $t \to \infty$,

$$\begin{aligned}
\mathbb{Q}\{T_a \leq t\} &= \mathbb{E}[e^{\gamma a - \frac{1}{2}\gamma^2 T_a}\underbrace{\mathbb{1}_{\{T_a \leq \infty\}}}_{=\,1\ \mathbb{P}\text{-a.s.}}] \\
&= e^{\gamma a}\underbrace{\mathbb{E}\left[e^{-\frac{1}{2}\gamma^2 T_a}\right]}_{=\,e^{-|\gamma||a|}} \\
&= \begin{cases} 1 & \text{if } \gamma a \geq 0\,; \\ e^{2\gamma a} & \text{else.} \end{cases}
\end{aligned}$$

7 Stochastic differential equations

7.1 Strong and weak solutions

Definition 7.1 (Stochastic differential equations)**.** Let $d, m \geq 1$, the applications

$$\begin{aligned}
\sigma : \mathbb{R}_+ \times \mathbb{R}^d &\to \mathcal{M}_{d,m}(\mathbb{R})\,; \\
b : \mathbb{R}_+ \times \mathbb{R}^d &\to \mathbb{R}^d.
\end{aligned}$$

are measurable and locally bounded. We define the SDE $E(\sigma, b)$

$$\mathrm{d}X_t \;=\; \sigma(t, X_t)\,\mathrm{d}B_t \;+\; b(t, X_t)\,\mathrm{d}t.$$

We say that $E(\sigma, b)$ has a solution if there exists:

- $(\Omega, \mathcal{F}(\mathcal{F}_t), \mathbb{P})$ filtered space,
- $B = (B^1, \ldots, B^m)$ $(\mathcal{F}_t)$-brownian motion $\mathbb{R}^m$-valued,
- $X = (X^1, \ldots, X^d)$ continuous adapted process s.t.

$$X_t \;=\; X_0 \;+\; \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s \;+\; \int_0^t b(s, X_s)\,\mathrm{d}s.$$

In the special case where $X_0 = x \in \mathbb{R}^d$, we say that $(\Omega, \mathcal{F}(\mathcal{F}_t), \mathbb{P}, B, X)$ is a solution of $E_x(\sigma, b)$.

Definition 7.2. (i) The weak existence of $E(\sigma, b)$ means that for all $x \in \mathbb{R}^d$, there exists a solution for $E_x(\sigma, b)$.

[11] This result is given by the theorem of Kolmogorov.

(ii) The weak uniqueness for $E(\sigma, b)$ means that $\forall x \in \mathbb{R}^d$, all the solutions for $E_x(\sigma, b)$ have the same law.

Example 7.1. Let $\mathrm{d}X_t = \mathrm{sgn}(X_t)\,\mathrm{d}B_t$. For a fixed $x \in \mathbb{R}$, $\forall(\Omega, \mathcal{F}(\mathcal{F}_t), \mathbb{P})$, β a $(\mathcal{F}_t)$-brownian motion, $\beta_0 = x$. And we define

$$B_t \;:=\; \int_0^t \mathrm{sgn}(\beta_s)\,\mathrm{d}\beta_s.$$

Then $\langle B \rangle_t = \int_0^t 1\,\mathrm{d}s = t$, and by Levy B is a $(\mathcal{F}_t)$-brownian motion. And then

$$\begin{aligned} & \mathrm{d}B_t \;=\; \mathrm{sgn}(\beta_t)\,\mathrm{d}\beta_t \\ \Leftrightarrow\quad & \mathrm{d}\beta_t \;=\; \mathrm{sgn}(\beta_t)\,\mathrm{d}B_t. \end{aligned}$$

Example 7.2. Let $\mathrm{d}X_t = \mathrm{d}B_t + b(t, X_t)\,\mathrm{d}t$. For all $(\Omega, \mathcal{F}(\mathcal{F}_t), \mathbb{P})$, $\forall X$ $(\mathcal{F}_t)$-brownian motion, let $\mathbb{Q}$ be a probability measure s.t.

$$\mathbb{Q}_{|\mathcal{F}_t} \;=\; \exp\left(\int_0^t b(s, X_s)\,\mathrm{d}X_s - \frac{1}{2}\int_0^t b(s, X_s)^2\,\mathrm{d}s\right)\cdot \mathbb{P}_{|\mathcal{F}_t}$$

so with Girsanov, $B_t := X_t - \int_0^t b(s, X_s)\,\mathrm{d}s$ is a $(\mathcal{F}_t)$-brownian motion under $\mathbb{Q}$. So $(\Omega, \mathcal{F}(\mathcal{F}_t), \mathbb{Q}, B, X)$ is a solution.

Definition 7.3. (i) Strong uniqueness for $E(\sigma, b)$ if two solution X and $\tilde{X}$ associated to the same filtered space and the same brownian motion s.t. $X_0 = \tilde{X}_0$ a.s. are indistinguishable.

(ii) We fix $(\Omega, \mathcal{F}(\mathcal{F}_t), \mathbb{P})$ and B $(\mathcal{F}_t)$-brownian motion. We say that X is a strong solution for $E(\sigma, b)$ if X is adapted according to the canonical filtration of B.

Example 7.3. We retake the SDE $\mathrm{d}X_t = \mathrm{sgn}(X_t)\,\mathrm{d}B_t$. There is no strong uniqueness, indeed if X is solution with $X_0 = 0$, then $-X$ is also solution.

Example 7.4. Consider the SDE

$$\mathrm{d}X_t \;=\; \lambda X_t\,\mathrm{d}B_t\,,$$

with $\lambda \in \mathbb{R}$. We know from Theorem 6.2 that there is strong uniqueness and for all x the solution is $X_t = x\exp\left(\lambda B_t - \frac{1}{2}\lambda^2 t\right)$.

Theorem 7.1 (Yamada–Watanabe)**.** *If there is weak existence and strong uniqueness then there is weak uniqueness.*